A novel computational framework for Electrical Impedance Tomography

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Abstract— Electrical Impedance Tomography is an emerging imaging modality, where boundary data (e.g. surface voltages) are used to image (interior) passive electromagnetic properties, i.e., conductivity and permittivity, of a body. In view of modern literature, EIT is classified as a promising portable, inextensive, and fast means of functional imaging. In this research, we propose a new framework based on B-Splines for the discretisations of the forward problem in EIT, due to their attracting features and inherent polynomial simplicity. The associated challenge however, is that B-Splines can be defined in polygonal domains at most, limiting their applicability. In order to overcome this issue, we investigate the use of Domain Embedding Methods suitably configured to result in a mesh-free formulation. This is of crucial importance to EIT as it overcomes the tedious and time consuming re-meshing procedure.

Keywords-Electrical Impedance Tomography,B-splines, domain embedding methods.

I. INTRODUCTION

EIT along with Electrical Capacitance Tomography (ECT), Magnetic Induction Tomography (MIT) and Magnetic Permeability Tomography (MPT) form a family of non-invasive and non-intrusive soft-field electromagnetic (ET) imaging techniques (see [6], [5] and references therein for a comprehesive review). Conceptually, the same, low frequency, biologically safe, operational strategy, is applied in order to generate an image; a known stimulation is applied to the boundary of the body and the developed (surface) fields are recorded, serving as the observable input to extract the unobservable distribution x.

What differs between the methods however, is the individual source of excitation on the boundary and the type of information x one may recover from each modality. For instance, in EIT a sequence of alternating current patterns $i:=Y$ (typically below 100KHz) is applied through some electrodes to the surface of the body and sets of the de-veloped boundary potentials $u:=U$ are recorded in some others.

On the other hand, since the applied excitation patterns are of low frequency and amplitude, the resulting fields are expected to hold weak values, which does not necessarily pose a problem in a simulation environment. Under a realistic data collection protocol however, measurements are highly contaminated with noise originating from different sources, in contrast to the well-known case of X-ray Computed Tomography (CT) applications belonging to the class of 'Hard-Field' imaging.

In CT, an ionising beam is used as the excitation pattern. Due to the high operating frequency (10^{15} 10^{18} Hz), the driving beam sinks in and out of the body with very little attenuation paving the way for high Signal-to-Noise-Ratio (SNR) observations independent from (internal) body variations, obviously, at the cost of the (long-term) harmful side-effects of the method. In contrast, the non-ionizing, 'non-invasive' nature of low-frequency driving patterns used in EIT cannot be confined to a plane and thus 'diffused' excitations are applied resulting, in turn, to weaker boundary fields.

That said, an internal variation (perturbation) either close or far away from the measurement pair can indeed affect not only the near-by data acquisition process but in fact all readings. In effect, given the 'diffusion' in measurements, one shall seek for changes off-the-driving-plane, a claim implying more indicative readings and reconstructions under 3D models and 3D measurement protocols. Therefore, an accurate geometric model is of paramount importance for EIT. In fact, model misfits are known to generate significant artefacts in the reconstructed images.

The simplicity of FEM, for the solution of the forward problem modelling arising from non-standard, complex geometries, coupled with some additional desired numerical features e.g., sparsity, can hardly be beaten in moderate imaging systems. The main drawback of FEM however, is the laborious and time-consuming (re-)meshing procedure which poses a fundamental problem when it comes to large scale and evolving domains. The key contribution of this research is the derivation of a mesh-free alternative framework to the conventional FEM forward formulation of EIT, which preserves all the desired properties of the FEM

approach, as for instance sparsity, whilst offering some additional advantages e.g. a mesh-free paradigm.

In the next section, we give a quick overview of the imaging principles of EIT. This is followed by the crucial forward problem modelling. Next, Domain Embedding Methods (DEM) are introduced to accommodate (potential) boundary variations and Spline basis function discretisation. A mesh free formulation can be obtained and condition numbers for the system matrix are presented. We conclude this study by addressing the condition number results and by discussing future work.

II. BACKGROUND

In an abstract soft-field imaging framework, given the assumed electromagnetic stimulations (e.g., current), a known initial distribution x (e.g., conductivity), interior fields (e.g., potentials) and surface measurements y (e.g., voltages), the image formation process can be summarized by means of a non-linear operator

$$\Lambda(x) = y \qquad (1)$$

where $\Lambda: X \to Y$ and X, Y are some Hilbert spaces. A 'forward problem' can be stated as given $x \in X$ is determine $y \in Y$. In a reverse manner, one may define an 'inverse problem' in which $x \in X$ is inferred from knowledge of $y \in Y$. Of particular importance is the determination of x when only partial knowledge of the full database y is accessible as there is no immediate promise about successful recover (see [3] for more sophisticated reconstructions).

Assuming that Λ is continuously differentiable, the simplest possible approach in order to extract x from Λ is to linearise Λ around a known point x_0 and to ignore higher order terms, effectively resulting to the familiar formula

$$\Lambda^{(1)}(x_0)\delta x = \delta y \qquad (2)$$

where

$$\delta x = x - x_0 \qquad (3)$$

and

$$\delta y = y - y_0,\ y_0 := \Lambda(x_0) \qquad (4)$$

and $\Lambda^{(1)}(x_0)$ is the first-order derivative of Λ evaluated at x_0.

Although this is hardly the case, assuming for convenience that $\Lambda(x_0)$ is invertible (in the truncated Singular Value Decomposition sense), the simplest possible inverse imaging problem can be derived as

$$\delta x = \left(\Lambda^{(1)}(x_0)\right)^{-1}\delta y \qquad (5)$$

In this respect, in order to image a body under an inverse problem formulation, that is to recover x from y, a simple multiplication of the inverse (linearised) forward operator with a set of measurements y suffices to estimate the unknown distribution x. As such the core of the EIT imaging problem boils down to a 'precise forward operator' and eventually forward modelling.

III. MODELLING: THE FORWARD EIT PROBLEM

Without loss of generality, assume a bounded domain $\Omega \subset \Pi \subset R^2$ where L electrodes are attached on its Lipschitz boundary surface $\partial\Omega$ [7]. $\Gamma \subset \partial\Omega$ denotes the union of areas under each electrode, assumed to be open connected subsets

$$\bigcup_{l=1}^{L}\Gamma_l = \Gamma \qquad (6)$$

whose closures are disjoint,

$$\bigcap_{l=1}^{L}\overline{\Gamma_l} = 0 \qquad (7)$$

And $\Theta := \partial\Omega \setminus \Gamma$ is the union of the remaining areas.

Assuming low-frequency stimulations patterns, Maxwell's equations can be simplified to a generalised Laplacian of the form

$$\nabla \cdot (\sigma \nabla u) = 0 \qquad (8)$$

Boundary conditions on Γ are defined as

$$\sigma \nabla u \cdot v = i \quad on\ \Gamma \qquad (9)$$

$$u + z_l \sigma \nabla u \cdot v = U_l \quad on\ \Gamma_l \qquad (10)$$

where σ, u, U_l, v, I, z_l are the admittivity, interior potential distribution, surface potential on the l-th electrode, the outward unit normal vector, current density and surface impedance, respectively. Note that on the inter-electrode gaps we assume no flux

$$\sigma \nabla u \cdot v = 0 \quad on\ \Theta \qquad (11)$$

Multiplying (8) with a test function $\overline{\omega}$ and integrating over the Ω (along with the application of vector identities and the divergence theorem) yields

$$\int_{\Omega} \sigma \nabla u \cdot \nabla \overline{\omega} d\Omega = \int_{\partial\Omega} \sigma \nabla u \cdot v \overline{\omega} d\Gamma \qquad (12)$$

Plugging in boundary conditions (1) and (11) in (12) yields

$$\int_{\Omega} \sigma \nabla u \cdot \nabla \overline{\omega} d\Omega = \sum_{l=1}^{L} \int_{\Gamma} \frac{1}{z_l} U_l \overline{\omega} d\Gamma_l - \sum_{l=1}^{L} \int_{\Gamma} \frac{1}{z_l} u \overline{\omega} d\Gamma_l \qquad (13)$$

On the other hand, the boundary conditions for the current density (9) require that

$$\int_{\Gamma} \sigma \nabla u \cdot v d\Gamma = \int_{\Gamma} j d\Gamma \qquad (14)$$

Or electrode-wise using (10)

$$\int_{\Gamma_l} \frac{1}{z_l}(U_l - u)d\Gamma = Y_l \quad (15)$$

Now considering the sesquilinear form [7]

$$a_\Omega((\upsilon,V),(\omega,W)) := \int_\Omega \sigma\nabla\upsilon\cdot\nabla\overline{\omega}d\Omega + \sum_{l=1}^{L}\int_{\Gamma_l}\frac{1}{z_l}(\upsilon - V_l)(\overline{\omega} - \overline{W_l})d\Gamma_l \quad (16)$$

As such, the weak formulation of the EIT problem on the original domain Ω can be stated as follows: Given the stimulation patterns (right hand sides) $Y=(Y_1, \ldots, Y_L)^T \in R^L$, with Y_L denoting the current applied to the l-th electrode, fund $(u, U) \in H^1_\Omega$ such that

$$a((u,U),(\upsilon,V)) = \sum_{l=1}^{L} Y_l\overline{V_l}, \quad for\ all\ (\upsilon,V)\in H^1_\Omega \quad (17)$$

where $H^1_\Omega = H^1(\Omega)\oplus C^L$ is the associated Sobolev space and $(H^1_\Omega)'$ denotes its dual.

Equation (17) is known to satisfy existence and uniqueness requirements as long as $\sum_l U_l = 0$ and $\sum_l Y_l = 0$.

IV. FICTITIOUS DOMAIN Π

A desired feature in many FEM oriented numerical schemes is to bypass the meshing process and come up with a mesh-free formulation. A typical example is the Finite Difference Method (FDM), where uniform grid discretisation is applied. Practically, however, FDM is limited to very simple rectangular domains, far from the ones encountered in EIT.

Alternatively, one may derive appropriate forms to map the original domain to a more numerically appealing one. Such formulations appear to tick the above requirements, apart from accommodating evolving boundaries, which is, for instance, appropriate in the case of continuous patient monitoring.

In this research, we opt for a mathematically challenging methodology, however, capable of fulfilling the problem specifications, by means of Domain Embedding Methods: the original domain Ω is extended to a larger one, yet of a simpler shape, in such a way that antipated perturbations of the evolving boundary surface can still be registered in the new 'fictitious' domain denoted as Π [4]. In view of the use of Spline functions and mesh-free methods, an appropriate methodology is to opt for a suitable square domain for Π (as in the case of Finite Differences).

A. Mathematical equivalence

The crucial part of this work is to ensure that the solutions for Π and Ω are equivalent. Taking suitable extensions for all functions from Ω to Π, the sesquilinear form in Π, denoted as $a_\Pi(\cdot,\cdot)$ is defined as

$$a_\Pi((\upsilon,V),(\omega,W)) := \int_\Pi \sigma\nabla\upsilon\cdot\nabla\overline{\omega}d\Pi + \sum_{l=1}^{L}\int_{\Gamma_l}\frac{1}{z_l}(\upsilon - V_l)(\overline{\omega} - \overline{W_l})d\Gamma_l \quad (18)$$

By comparing (16) and (18), it may appear that the sesquilinear forms are equivalent. However, this is not the case. The theoretical pitfall here originates from the fact that in the (inter-electrode gaps Θ), the flux condition (11) in the original domain Ω is implicitly encapsulated in a_Ω (but vanishes). The same condition is not fulfilled by a_Π. In other words, there is no restriction about the flux of the potentials in the new fictitious domain Π.

In mathematical terminology, the natural boundary conditions on Ω translate to an essential boundary condition in Π. Such a claim entails that if equivalence between the two domains is desired, one needs to explicitly enforce the flux condition in the gaps Θ between the electrodes.

B. Enforcing zero flux on the interelectrode gaps

We opt for the Langrange multiplier technique to enforce the essential boundary condition in the fictitious domain Π by setting the appropriate functional

$$\inf_{(\upsilon,V)\in H^1_\Pi}\sup_{q\in(H^{1/2}(\Theta))'}\{\frac{1}{2}a_\Pi((\upsilon,V),(\upsilon,V)) - \sum_{l=1}^{L}Y_l\overline{V_l} + b(\upsilon,q)\} \quad (19)$$

where

$$b(\upsilon,q) := \int_\Theta q\sigma\nabla\upsilon\cdot\nu ds \quad (20)$$

Consider now the standard first order optimality conditions, where the standard saddle point formulation can be considered as

$$a_\Pi((u,U),(\upsilon,V)) + b(\upsilon,p) = \sum_{l=1}^{L}Y_l\overline{V_l} \quad (21)$$

$$b(u,q) = 0 \quad (22)$$

These give rise to an equivalent operator form system as

$$\begin{bmatrix} A & B' \\ B & 0 \end{bmatrix}\begin{bmatrix} [u,U]^T \\ p \end{bmatrix} = \begin{bmatrix} \sum_{l=1}^{L}Y_l\overline{V_l} \\ 0 \end{bmatrix} \quad (23)$$

where *A:* $H^1_\Pi \rightarrow (H^1_\Pi)'$ is a self-adjoint (domain) operator and *B:* $H^1(\Pi) \rightarrow (H^{-1/2}(\Theta))'$ is the appropriate boundary (trace) operator. Note that we denote its adjoint by *B'*.

C. Existence and Uniqueness

For existence and uniqueness, one needs to demonstrate continuity and coercivity for the operator *A*, which is trivially known by recalling [7]. However, demonstrating that the operator *B* fullfils the necessary inf-sup conditions is mathematically more involved and out of the scope of the current contribution. The actual proof of existence and uniqueness in domain Π is established and will be published in a future contribution.

V. LINEAR B-SPLINE DISCRETISATIONS

Having derived the DEM formulation for the EIT problem in the continuous domain, we now proceed with Spline discretisation. In broad terms, Splines φ are Riesz polynomial basis functions exhibiting compact support. The functions considered in this work are piecewise linear hat functions.

Assuming a finite subspace $S_j := S(\varphi) \subset L_2$ the EIT functions can be assumed to be linear combinations of Spline functions as

$$u(x) = \sum_k c_k \varphi(\frac{x}{2^j} - k) \qquad (24)$$

$$v(x) = \varphi(\frac{x}{2^j} - k) \qquad (25)$$

where $h = 2^{-j}$. The sesquilinear form Π can now be discretised as

$$a_\Pi((u,U),(\omega,W)) = a_\Pi((\sum_k c_k \varphi(\frac{x}{2^j} - k),1),(\varphi(\frac{x}{2^j} - k),1)) \qquad (26)$$

The discretisation of $b(\cdot,\cdot)$ is somewhat more involved. Assuming that the complex conductivity σ (admittivity) near Π is fixed (e.g., tissue under the skin), one obtains

$$b(v,q) \approx \int_\Theta q \nabla v \cdot v ds \qquad (27)$$

Assuming traces for the domain functions and suitable Spline-boundary functions [2], denoted as $\varphi_\theta(x) := \varphi(\frac{x}{2^j} - m)$, the sesquilinear form $b(\cdot,\cdot)$ can be discretised

$$b(v,q) = b(\varphi(\frac{x}{2^j} - k), \varphi(\frac{x}{2^j} - m)) \qquad (28)$$

The uniform grid discretisation concept perfectly fits the use of Splines. Additional advantages when employing these functions is that domain integrals and correspondingly, the associated derivatives have fixed, non-zero entries, which can be pre-calculated. This enables a banded structure for the system matrix as opposed to scattered matrix entries in typical FEM matrices. Note that the above are independent of the linear B-splines chosen here and hold valied for higher-order B-Splines, if desired. The computation of contour intergrals however is more involved, and as in FEM, requires a mapping to a specific interval.

A. Domain Discretisation

For functions defined in the domain Π, discretisation can be performed in a rather trivial way due to the square (fictitious) domain, allowing for uniform grid discretisation (discretisation pitch *h*). Naturally, we opt for a square fictitious domain of unit side for computational convenience. Next, we register in Π, a circular domain (Ω) of radius 1/4, centred at (0.5, 0.5) with 4 electrodes attached, holding fixed contact impedance values set to *z=10 Ohm*.

Note that since the Laplacian operator is singular, operator *A* is anticipated to hold arbitrary high condition numbers. In order to address this deficiency, we need to include a boundary condition for $\partial\Pi$. We will refer to these conditions as c_0. The simplest setting is to assume zero boundary condition on $\partial\Pi$. Note that as long as the radius of the original domain is not overlapping with $\partial\Pi$, the approach is valid. The actual determination of the optimal distance between the original and the fictitious domains is a subject for further study. In practice, however, one assumes some distance ε from $\partial\Pi$ such that $h>>\varepsilon$.

B. Boundary Discretisation

The discretisation of the boundary contour is a delicate issue. Following the dyadic discretisation of the domain Π, we opt for a dyadic discretisation $\partial\Omega$ by splitting the boundary curve into an equivalent number of linear elements. Given the c_0 boundary conditions, the domain functions are equivalently discretised with zero boundary conditions. Having said this, for the boundary functions φ_θ, there are two options. One could assume zero boundary functions or special boundary functions with adapted boundary conditions as in [2].

TABLE I. CONDITION NUMBERS OF **A** WITH C_0 BOUNDARY CONDITIONS

j	*A*
3	***1.76e+001***
4	4.35e+000

TABLE II. CONDITION NUMBERS OF **B** WITH C_{BC} AND C_0 BOUNDARY CONDITIONS, ξ=3

l	B_{BC}	B_0
01	***3.13e+000***	***2.39e+000***
02	2.71e+000	2.71e+000
03	3.14e+000	2.39e+000
04	2.71e+000	2.71e+000

VI. NUMERICAL RESULTS

As iterative solvers are employed in EIT, it is of primary interest to report on the associated condition numbers. The condition number is determined as the ratio of the largest to the smallest singular value and practically is of paramount importance for any kind of iterative solver; the smaller the condition number, the faster the convergence of the iterative scheme. Thus, we focus our numerical results in the determination of the condition numbers for the various configurations.

TABLE III. CONDITION NUMBERS OF **B** WITH C_{BC} AND C_O BOUNDARY CONDITIONS, ξ=4

l	B_{BC}	B_0
01	***1.39e+001***	***7.28e+000***
02	5.19e+000	5.09e+000
03	1.43e+001	7.30e+000
04	5.18e+000	5.08e+000

TABLE IV. CONDITION NUMBERS OF **B** WITH C_{BC} AND C_O BOUNDARY CONDITIONS, ξ=5

l	B_{BC}	B_0
01	***2.82e+016***	***1.69e+003***
02	1.42e+016	1.20e+002
03	2.29e+016	4.10e+003
04	1.35e+017	1.20e+002

The condition numbers for the discrete counterparts of operators A and B, denoted as **A** and **B** are computed and presented in Tables I-IV. We use an iterative saddle point solver (e.g., Uzawa) to calculate A and B. The condition numberof matrix A holds a steady growth, which is proportional to the discretisation level j. The condition number of matrix B appears to be less trivial to interpret. This is due to the so called LLB condition originating from the inf-sup condition that B must meet (see [1], [4]). In simple terms, this accounts to different discretisation levels, denoted as ξ, for the spaces associated with the functions defined on the boundary, rather than the traces of the ones defined on the boundary. It turns out that the condition number maintains a low value as long as $\xi<j$. This implies the use of a smaller pitch on Π for a finer discretisation on the fictitious domain than the one used to discretise the boundary contour. In the results presented, the values are j=06, ξ=3, 4, 5 for Π and Θ, respectively.

The simplicity of the proposed formulation can easily be ruined if care is not exercised in the numerical discretisation. This can lead to misinterpretations of the results and numerical instability. Hence, it is crucial to study the discretisation effects by means of condition numbers. This is in line with the goals of the application at hand, i.e., EIT.

VII. CONCLUSIONS AND FURTHER WORK

In this research, the forward problem in EIT was considered. The main imaging principles were addressed and the importance of the forward problem became apparent. The weak formulation of the forward problem was derived and DEM were considered for EIT. The suggested formulation was presented in the continuous space and the theoretical equivalence between original and fictitious domains was addressed. Discretisation was performed by means of linear B-Splines. Increasing condition numbers were reported, in line with theoretical studies.

Since splines admit multi-resolution, it would be natural to extend this formulation to accommodate wavelet functions. Such a study would be rather interesting due to the inherent preconditioning properties of wavelets. This approach can also be extended to other imaging modalities.

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