Delay and Doppler Estimation of Gaussian Mixtures Using Moment

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Abstract— In this paper, a new moment-based method for joint time delay and Doppler estimation in passive radar, sonar, and GPS applications is described. We provide a new blind approach for estimating the delay and Doppler imposed onto a Gaussian mixture using on the statistical moments. Wigner-Ville (WV) distribution has suitable estimation for delay and Doppler. However, since WV is a bilinear transformation, it suffers from the quadratic superposition principle, i.e. cross-term problem. Different attempts for cross-term suppression resulted in numerical complexity. .Hence, there is a trade off between suppression and complexity. A new method is discussed for resolving this issue. The estimation is done by using the moments of received signal. This procedure provides lower complexity and more accuracy. In this method, the noise power is assumed unknown and is estimated as well. The simulation results demonstrate that the proposed approach can effectively estimate the delay and Doppler in noisy environment.

Keywords- delay and Doppler estimation; moment generating function; non-stationary signals; noise modeling.

I. Introduction

In radar, sonar, and Global Positioning Systems (GPS), the time delay and Doppler are two important parameters that facilitate the target localization and tracking [1]. WV has proven to be a valuable tool in time-frequency processing. WV possesses a high resolution in the time-frequency plane, and satisfies a large number of desirable theoretical properties [2]. These properties are in fact the fundamental motivation for the use of the narrowband(wideband) WV transformation for detecting a deterministic signal with unknown delay-Doppler(-scale) parameters in white Gaussian noise. Unfortunately, WV's practical usage is limited by the presence of non-negligible cross-terms, resulting from interactions between signal components.

Cross-WV distribution of two signals $x(t)$ and $y(t)$ is defined as:

$$W_{x,y} = \int_{-\infty}^{\infty} x(t+\tau/2)\, y^{*}(t+\tau/2)\exp(-j2\pi f\,\tau)d\tau. \quad (1)$$

When time and frequency details of a signal is prominent, this distribution presents an accurate response, but, when one of these two signals is affected by the noise, WV distribution could not effectively distinguish them due to the presence of cross-terms. Alternative approaches are proposed for eliminating or at least, suppressing the cross-terms [2]–[5]. Generally speaking, the cross-term suppression may be divided into two categories: signal-independent, and signal-dependent paradigm. Coupling the Gabor transformation with the WV distribution is a signal-independent procedure that reveals a cross-term suppression approach through the exploitation of partial knowledge about the signals to be encountered [3]. For the signal-dependent method, it is possible to apply an adaptive window over WV distribution, where the kernel parameters are determined automatically from the parameters of the input signal. This kernel is capable of suppressing the cross-terms and maintaining accurate time-frequency resolution [4]. It is not possible to generate an alias-free discrete WV distribution from a discrete analytic signal, but, there is a new discrete analytic signal that indirectly suppresses cross-term [5].

In addition to WV method, there are other techniques for the delay and Doppler estimation. Wavelet transform is one of these procedures. Wavelet approach combines the noise filtering and scaling together, yielding a reduction in complexity [6]. There is also another method which uses the fractional lower order ambiguity function (FLOAF) for joint delay and Doppler estimation [7]. In this paper, we present another view for solving the problem which has less complexity and also suppresses cross-term effectively. It is assumed that the transmitted signal follows an *N*-mode Gaussian mixture model (GMM). GMM can be used for different transmitted signals. Especially, it presents an accurate modeling for actual signals transmitted in sonar and radar systems [8]. The received signal is contaminated by noise, delay and Doppler. It is shown that how it is possible to distinguish signal and noise by using PDF of received signal and moments which have been extracted from this PDF. Having noise statistics is equivalent to cross-term suppression of multi-component signals in Wigner-Ville distribution discussions. Then, the delay and Doppler will be attained by using the results obtained from the available moments.

Section 2 provides a model for the received signal. This signal has been influenced by unknown noise, delay and Doppler. It is shown in Section 3 that it is possible to estimate Doppler and the unknown noise power simultaneously by using the moments of the received random signal. There is also a method for the delay estimation based on the PDF of the received signal which is described at the end of this section. Section 4 contains simulation results to illustrate the effectiveness of the proposed method.

II. Signal Model

Mixture models have been proven to be quite useful in modeling complex densities [9]. Using a small number of normal components, one is able to model distributions that are far from normal.

A signal transmitted from a remote source and monitored in the presence of the noise along with a relative motion between the transmitter and the receiver can be modeled as:

$$y(t) = s(t-\tau)\exp(j2\pi t\,\varepsilon) + \omega(t), \quad (2)$$

where $s(\cdot)$ is the transmitted signal which is modeled at each time instance t to follow a real N-mode Gaussian mixture:

$$\sum_{i=1}^{N} p_i \mathrm{N}(\mu_{s_i}, \sigma_{s_i}^2), \tag{3}$$

and ω is a real zero-mean additive white Gaussian noise (AWGN) with power of σ_ω^2. The signal and noise are assumed to be uncorrelated. τ and ε denote the delay and Doppler respectively.

III. MOMENTS

Let's assume that $f_X(x)$ is the PDF of the random variable X whose moments are given as:

$$m_n = \int_{-\infty}^{\infty} x^n dF(x) = E(x^n), \tag{4}$$

where F is the cumulative density function (CDF) of random variable X and E describes the expectation value of the variable. On the other hand, moment generating function (MGF) of the random variable X is defined as:

$$M_x(u) = E(e^{uX}), \quad u \in \mathbb{R}, \tag{5}$$

whenever this expectation exists. The relation between moments and MGF is used in this paper:

$$M_x(u) = 1 + um_1 + \frac{u^2 m_2}{2!} + \cdots. \tag{6}$$

This derivation is valid as long as moments m_n are finite, $|m_n| < \infty$.

A. Doppler estimation

The statistical properties of the signal and noise are known in (2), therefore, their MGF is available. Although the transmitted signal follows a Gaussian mixture distribution, the conglomerate effect of the delay and Doppler creates a non-stationary signal. Doppler estimation is described here and the discussions about the delay estimation are provided in the sequel. First, it is required to consider the normal distribution as the base for the next steps. Its MGF is:

$$M_x(u) = \exp(\mu u + \frac{\sigma^2 u^2}{2}), \tag{7}$$

where μ and σ^2 are the mean and variance of normal distribution respectively. Its moments can be easily achieved. Suppose that the received noise free signal is denoted by $r(t)$. Since $y(t) = r(t) + \omega(t)$, and $r(t)$ and $\omega(t)$ are independent processes, MGF for $y(t)$ is:

$$M_y(u) = M_r(u) M_\omega(u), \quad M_\omega(u) = e^{0.5\sigma_\omega^2 \mu^2}, \tag{8}$$

$\omega(t)$ is a normal variate. The problem is to calculate MGF of $r(t)$. Without loss of generality, let's assume that we have no delay, thus:

$$r(t) = s(t)\exp(j 2\pi t \varepsilon). \tag{9}$$

TABLE I. MOMENTS OF RANDOM VARIABLE r_r

n	*Moments*
0	1
1	$\sum_{i=1}^{N} p_i \mu_{s_i}$
2	$\sum_{i=1}^{N} p_i (\mu_{s_i}^2 + \sigma_{s_i}^2 \cos^2(2\pi t\varepsilon))$
3	$\sum_{i=1}^{N} p_i (\mu_{s_i}^3 + 3\mu_{s_i} \sigma_{s_i}^2 \cos^2(2\pi t\varepsilon))$
4	$\sum_{i=1}^{N} p_i (\mu_{s_i}^4 + 6\mu_{s_i}^2 \sigma_{s_i}^2 \cos^2(2\pi t\varepsilon) + 3\sigma_{s_i}^4 \cos^4(2\pi t\varepsilon))$
5	$\sum_{i=1}^{N} p_i (\mu_{s_i}^5 + 10\mu_{s_i}^3 \sigma_{s_i}^2 \cos^2(2\pi t\varepsilon) + 15\mu_{s_i} \sigma_{s_i}^4 \cos^4(2\pi t\varepsilon))$

The goal is the estimation of σ_r^2. For simplicity, we work with the real part of $r(t)$:

$$r_r(t) = s(t)\cos(2\pi t\varepsilon). \tag{10}$$

$s(t)$ follows a Gaussian mixture distribution in (3), but, the presence of the cosine term in (10) changes $r_r(t)$ to a non-stationary process. Although this term is time variant, fortunately, it is deterministic. Now we obtain MGF of $r_r(t)$:

$$\begin{aligned} M_s(u) &= \sum_{i=1}^{N} p_i \exp(\mu_{s_i} u + 0.5\sigma_{s_i}^2 u^2) \\ M_{r_r}(u;t) &= \sum_{i=1}^{N} p_i \exp(\mu_{s_i} u + 0.5\sigma_{s_i}^2 \cos^2(2\pi t\varepsilon) u^2). \end{aligned} \tag{11}$$

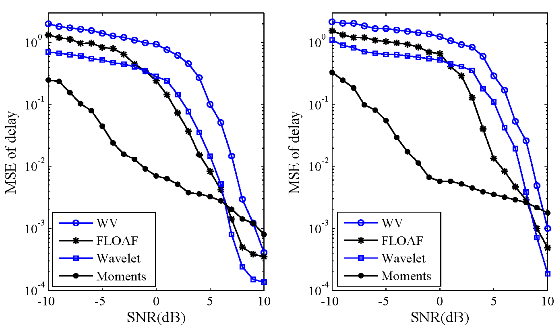
According to (6), the moments of the random variable r_r could be calculated. These moments are depicted in table I.

Now, the moments of the received signal $y(t)$ can be obtained. Both MGF of $r_r(t)$ and $\omega(t)$ are expressed as the series presented in (6), then, by multiplying these two series and ordering their terms, MGF of $y(t)$ is obtained in context of (6):

$$\begin{aligned} &M_y(u;t) = M_{r_r}(u;t) M_\omega(u) = \\ &1 + u(m_{r_r1} + m_{\omega1}) + \frac{u^2(m_{r_r2} + m_{\omega2} + 2m_{r_r1} m_{\omega1})}{2!} \\ &+ \frac{u^3(m_{r_r3} + m_{\omega3} + 3m_{r_r1} m_{\omega2} + 3m_{r_r2} m_{\omega1})}{3!} \\ &+ \frac{u^4(m_{r_r4} + m_{\omega4} + 6m_{r_r2} m_{\omega2} + 4m_{r_r1} m_{\omega3} + 4m_{r_r3} m_{\omega1})}{4!} \\ &+ \cdots, \end{aligned} \tag{12}$$

which $m_{r_r i}$'s have been shown in table 1, and $m_{\omega i}$'s are the moments of a normal distribution. Therefore, the moments of the received signal $y(t)$ are obtained. The resulted moments are time dependent, but, since the cosine term is deterministic, the time average of the moments can be substituted instead. Let's define:

$$\zeta_i(\varepsilon) = \frac{1}{T}\int_0^T \cos^i(2\pi t\varepsilon) dt, \tag{13}$$



(a) (b)

Figure 1. MSE of delay estimation in conventional and proposed methods (a) constant noise variance. (b) variable noise variance.

which T is the time duration of the signal block.

Note that for the dependency of $\zeta_i(\varepsilon)$ on ε, the moments of the received signal are dependent on ε too. Finally, the time independent moments of $y(t)$ are as presented in (12), only all " $\cos^i(2\pi t\varepsilon)$ " terms should be substituted by $\zeta_i(\varepsilon)$. On the other hand, let's suppose that the moments of the observed signal in the receiver are calculated statistically by:

$$\mu_i' = \frac{1}{T}\int_0^T y^i(t)dt. \tag{14}$$

Both of these two procedures must yield same results. Thus, ε should be selected in such a way that this equality satisfies. To do this, the mean square error (MSE) estimator is used:

$$\text{MSE} = \sum_{i=1}^{m} \left|\mu_i - \mu_i'\right|^2, \tag{15}$$

where in this paper, m is considered 4, because it would reveal a desirable result. So Doppler of the received signal $y(t)$ is:

$$\hat{\varepsilon} = \min_{\varepsilon} \sum_{i=1}^{m} \left|\mu_i - \mu_i'\right|^2. \tag{16}$$

B. Noise estimation

The noise power estimation is similar to the Doppler estimation. Indeed, these two estimations are done simultaneously. In (12), it is seen that the moments do not merely depend on Doppler. They also depend onto the noise power as well. So, in (15), MSE includes two parameters and should be minimized according to both Doppler and the noise power of the received signal:

$$(\hat{\varepsilon}, \hat{\sigma}_\omega^2) = \min_{\varepsilon, \sigma_\omega^2} \sum_{i=1}^{m} \left|\mu_i - \mu_i'\right|^2. \tag{17}$$

Note that in the actual scenario, the noise variance in (8) is unknown. We can estimate the noise variance given N_1 signal free samples which are at hand occasionally, hence, σ_ω^2 becomes a random variate. Since the noise $\omega(t)$ is assumed Gaussian, the N_1-sample based estimated variance is chi-square distributed with N_1 degrees of freedom:

$$\hat{\sigma}_\omega^2 = \frac{1}{N_1}\sum_{i=1}^{N_1}\omega_i^2, \quad \hat{\sigma}_\omega^2 \sim \chi_{N_1}^2, \tag{18}$$

Hence, the average MGF of the noise over the sample variance is obtained:

$$\begin{aligned} \bar{M}_\omega(u) &= \frac{1}{\sqrt{(1-\hat{\sigma}_\omega^2 \mu^2 / N_1)^{N_1}}} \\ &= 1 + 0.5\hat{\sigma}_\omega^2 \mu^2 + (0.125 + 1/4N_1)\hat{\sigma}_\omega^4 \mu^4 + \cdots. \end{aligned} \tag{19}$$

If this MGF is used for the noise in (12), the equation (17) would estimate the variance of this non-stationary noise. In section 4, there are results for both situations, i.e. constant and variable variance.

C. Delay estimation

Before discussing about the last parameter, delay, at first it is better to talk about changes that happens to a signal when it is exposed at delay phenomena. In some papers, the correlation between the transmitted and the received signal is criteria for estimating the delay. But since possessing of the transmitted signal is not reasonable, there are two sensors in the receiver, and the delay estimation is performed based on the correlation between these two receivers [10]. In the OFDM systems, the existence of cyclic prefix magnifies this correlation about the delay-point. Therefore, the delay estimation becomes more comfortable [11].

In this paper, we take a new direction for this estimation. Instead of tracking the variation of the correlation, the effect of the delay on the PDF of signal is examined. In this method, there is no need to have two sensors. Suppose that there is a delay and the receiver is waiting for the signal during this delay time. Before sensing the signal, the receiver only receives the noise, which has constant statistics. As soon as the main signal is detected, the statistical properties of the received signal changes, because, there is no pure noise anymore, but the receiver is taking both the noise and the transmitted signal. This change is a reasonable criterion for estimating delay point. Even there is no requirement to have knowledge about the noise power. Only monotonousness of signal power before the arrival of transmitted signal, and its change after sensing the main signal is sufficient for detecting the delay. To detect this change, a rectangular window is used. The length of this window is variable. The onset of this window is always the beginning of signal and the PDF estimation is done on the windowed signal. The window length increases and the endpoint of window moves to the end of the signal, and after each change in window length, a bigger segment of the signal is used for PDF estimation. The variation of statistics, like mean and variance, help finding the delay point. Those segments, which have the most change relative to the previous ones, are determined. The first segment of them is the most favorite and the end point of that is considered as the amount of delay.

This delay estimation can be done at the first stage before the Doppler and noise estimation, but it is better to estimate the noise power first, because this operation helps getting a more accurate estimation for delay even in a very unsuitable noisy environment.

IV. Simulation And Results

In this section, the proposed moment method is simulated to assess its performance to estimate the delay and Doppler. The transmitted signal has tri-modal Gaussian mixture distribution

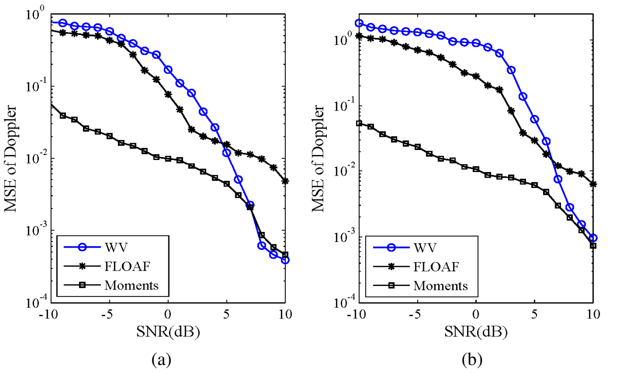


Figure 2. MSE of Doppler estimation in conventional and proposed methods (a) constant noise variance. (b) variable noise variance.

with $\sigma_{s_1}=\sigma_{s_2}=\sigma_{s_3}=1$, $\mu_{s_1}=2$, $\mu_{s_2}=5$, $\mu_{s_3}=8$, and $p_1=0.3$, $p_2=0.3$, $p_3=0.4$.

For each SNR, the simulation is performed 1000 times, the signal duration is 1msec, the delay is assumed 300μs, and the Doppler value, i.e. $\omega_\varepsilon=2\pi\varepsilon$, is a number between 0 and 2π that provides a 2π rotation for frequency shift. In the simulation, Doppler is assumed to be 0.8π.

Fig. 1 depicts the error existed in the estimation of delay for the conventional and proposed methods. This error is presented as MSE, calculated from 1000 times of simulation, versus SNR. Fig. 1(a) is related to constant noise variance and Fig. 1(b) shows MSE when noise variance is the random variable in (18). The presented conventional methods are WV in [2], wavelet method in [6] and FLOAF in [7]. It can be seen that in high SNR, the consequences of the proposed method have a little larger MSE relative to the other methods. It is possible to decrease this error by using more points in the PDF estimation, but it should be a trade off between the complexity and accuracy. Anyhow, this MSE is small and in comparison with the promising results that this method presents in low SNR, this error is acceptable. On the other hand, Fig. 1(b) depicts when we have a non-stationary noise, in addition to low SNRs, the proposed method also has better results in high SNR in comparison with the conventional methods.

There is a similar observation for Doppler. The related MSE is shown in Fig. 2. In this figure, the conventional methods are WV in [2] and FLOAF in [7]. As presented in above, Fig. 2(b) shows the proposed method has the best result relative to other methods for non-stationary noise.

It is worth mentioning that these results are in an unknown noise power scenario. To judge the accuracy of the proposed method for the estimation of the unknown noise power, a comparison between the actual and the estimated noise power is portrayed in Fig. 3. This comparison is presented as MSE of noise power estimation for both constant and variable variance.

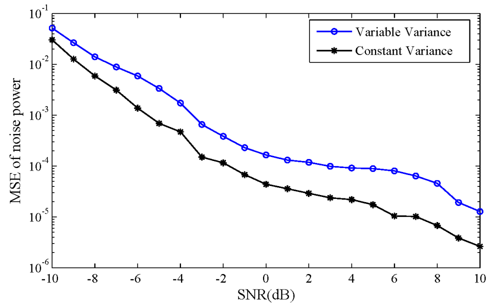


Figure 3. MSE of noise power estimation in proposed method.

V. Conclusion

In this paper, we provide a new approach to estimate the delay and Doppler imposed on a signal modeled as a Gaussian mixture. The new method is based on the moments of the received signal. This method is more successful than the Wigner-Ville method in environment which has unknown noise power. Another feature for this method is the simultaneous estimation of Doppler, and the noise power. About the delay estimation, there is no necessary need to attain noise nature before determining the delay, and even without knowing the value of the noise power, the predicted delay point has an acceptable accuracy. The simulation results show that these claims are trustable and present acceptable consequences.

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