USING WAVELETS ON DENOISING INFRARED MEDICAL IMAGES DATA BASE

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Introduction

- Computed aided diagnostic - CADx system makes substantial use of image processing and a great amount of data => efficient content based retrieval from image database

- Image restoration after storage and transition is fundamental for the quality of the other stages in the image processing.

- Studies showed that infrared (IR) based image analysis could identify breast modifications earlier than others exams.

- To be efficiently implemented, CADx must first consider a great number of patients followed by years; maintain record and comparison with others types of diagnoses, combine and integrate data to allow mining possible conclusions system.
Thermograms are acquired by a thermographic camera that is sensitive to infrared IR.

It is a physiological examination and is 50x cheaper than the mammogram.

IR has potential to detect breast cancer 10 years earlier than the nowadays traditionally golden method.

It can also be used for diagnosis of young women’s tumours (young breasts present dense tissues that makes difficult early detection of pathologies by the X-ray).
IR do not use ionizing radiation, venous access (or others invasive procedures), is painless and do not touch the patient.

The problem is the absence of CAD systems to aid the such diagnosis.

*Retroareolar Carcinoma*

You can see that this is a normal breast (very symmetrical!) but how to make the computer “see” the same?
CADx Pipeline

- thermal images
- automatic segment two regions of interest (ROI): the right breast and the left one;
- Then the extraction of the features;
- The last step is the classification of the patient image.
The project on development:
Features Extraction → Classification

48 features

24 features
AI and techniques of machine learning as used for classification

Breast reconstruction by IR images
Finite Volume Method for 3D modeling
Experimental x Numerical

- The depth and the radius of a hypothetical tumor are considered to calculate breast temperatures through parametric analysis.
Comparison between the temperature profiles of the thermogram and the surrogate breast meshed with 13748 nodes
Main objective:

- Best discrete wavelet (DW) scheme for
  - denoise,
  - storage and
  - retrieval

for the project of an infrared image database to aid breast disease diagnostic in a tropical climate country

ProEng project:

http://tvbrasil.ebc.com.br/reporterbrasil/video/31312/
First part

• Results and conclusions of an experimental study that intent to find the best family of wavelets to reduce noise of medium resolution infrared images.
8 different real images + noise

resolution: 640x 480

3 degradation levels
Additive White Gaussian Noise (AWGN):
\( \sigma = 5, \)
\( \sigma = 25, \)
and
\( \sigma = 50 \)

Total: 32 images of same type separated on 4 groups concerning the level of noise (0, 5, 25 and 50).
Low and High pass filters

DWT

Reconstruction

IDWT
Discrete wavelet transforms (DWT) is very effective in analyzing images because it at same time
reduce the storage,
improve the image quality and
promote content based retrieval of the data.

What is the best wavelet approach to be used in a project of an image database for medium resolution infrared images in screening of breast diseases.
Types experimented (with various vanishing moments)

- Biorthogonal: 1.1 to 6.8
- Coiflets, 1 to 5
- Daubechies, 2 to 45
- Haar,
- Meyer,
- Reverse Biorthogonal: 1.1 to 6.8
- Symmlets - 2 to 28

composing a total of 108 different variations!
Generic denoising procedures by DWT involve three steps:

- wavelet decomposition,
- threshold of coefficients related to noise in the wavelet domain, and
- reconstruction by inverse wavelet transform into the spatial domain.
Denoising

In/Original Image → Transform → Quantization → Coding → Out / Compressed Image

denoising by thresholding wavelet coefficient of detail
Decomposition & Denoising
wavelet decomposition step

• an image is decomposed into a sequence of spatial resolution images using DWT.

• In these, a given \( j \) level of decomposition can be performed resulting in \( 3^j + 1 \) different frequency bands of low (L) and high (H) components of the original image, namely, \( \text{LL}_j \), \( \text{LH}_j \), \( \text{HL}_j \) and \( \text{HH}_j \)
Wavelet denoising

- Identify low and high energy coefficients
- Modify noisy coefficients by adaptive thresholding
- We use the optimal reconstruction threshold:

\[ T = \frac{\sigma_n^2}{\sigma} \]

\[ \sigma_n^2 = \text{Noise variance} \]

\[ \sigma = \text{Original Signal variance} \]

(and Hard & soft Thresholding approach)
Setting to zero value of coefficients which are considered negligible.

\[
y_{soft}(t) = \begin{cases} 
\operatorname{sgn}(x(t) \cdot (|x(t)| - \delta)), & |x(t)| > \delta \\
0, & |x(t)| \leq \delta 
\end{cases}
\]

\[
y_{hard}(t) = \begin{cases} 
x(t), & |x(t)| > \delta \\
0, & |x(t)| \leq \delta 
\end{cases}
\]

where \( \delta \) is the threshold value, and \( \operatorname{sgn}(\ ) \) is the signal function (it results +1 when the argument is up to zero and -1 otherwise).
The thresholding method proposed is not based on a unique value $\delta$ for threshold but testing all possibilities for achieving better quality of the denoised image.

Values of threshold in a series of possibilities $\delta(n)$ are defined and related to each element $n$ of this series.

To consider the reconstructed image quality the normalized cross correlation (NCC) between the original and the denoised images is estimated.

In this case when more correlated are the images better is the $\delta$. 
Then the best threshold value for a given image is found automatically

- by the system considering best quality possible for the restored image when all others parameters are defined.

- Such search is put in an admissible computational time by using discrete possibilities previous delimited the best $\delta$ is found by a function of complexity $O(\log(n))$. 
Optimal reconstruction threshold

Example of the concave function relating threshold index and NCC for the best result of the base Biorthogonal 1.3.
Low and High pass filters

DWT

Reconstruction

IDWT
Decomposition on levels 3

- \((j=3)\) levels of high (H) and low (L) sub bands.

<table>
<thead>
<tr>
<th>LL^1</th>
<th>LH^3</th>
<th>LH^2</th>
<th>LH^1</th>
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<td></td>
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<td></td>
<td>HL^2</td>
<td>HH^1</td>
<td></td>
</tr>
<tr>
<td>HL^1</td>
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<td></td>
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</tr>
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</table>

1, 2, 3 --- Decomposition Levels

H ------ High Frequency Bands

L ------ Low Frequency Bands
2d

All lines → 1 line

Then columns

(a) Standard

(b) Pyramidal
Steps used on experiments with synthetic added noise images

Step 1: Image acquisition and storage as a raw data.

Step 2: Gaussian noise addition. Three levels of a standard deviation value ($\sigma_{\text{noise}} = 5, 25$ and $50$) are added.

Step 3: Define the type of wavelet, level of adaptive decomposition and the threshold process. Then the system select the coefficient for threshold based on the normalized cross correlation (NCC) that produces greater correlation.

Step 4: Image restoration

Step 5: Verification
Results

• For the 8 images, each of the 108 bases are tested for levels 3 and 4 of the decomposition (L3 and L4), and the 2 possible way of coefficient thresholding (soft and hard).

• Each configuration has been considered for the images with added Gaussian noise at three different levels, with the best thresholding value automatically computed, resulting in a total of 10368 experiments
For each configuration the evaluators are:

\[
RMSE = \sqrt{\frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} [G(x, y) - F(x, y)]^2}
\]  

(1)

\[
SNR_{ms} = \frac{\sum_{x=0}^{M-1} \sum_{y=0}^{N-1} G(x, y)^2}{\sum_{x=0}^{M-1} \sum_{y=0}^{N-1} [G(x, y) - F(x, y)]^2}
\]  

(2)

\[
SNR_{rms} = \sqrt{SNR_{ms}}
\]  

(3)

\[
PSNR = 20 \log_{10} \left( \frac{2^n - 1}{RMSE} \right)
\]  

(4)
### Used WT types

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<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
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<th>G</th>
<th>H</th>
<th>I</th>
<th>J</th>
<th>K</th>
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<td>Bior 3.7</td>
<td>db 1</td>
<td>db 11</td>
<td>db 21</td>
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<td>-</td>
</tr>
</tbody>
</table>


All images used are acquired by a Flir S45 camera (with sensitivity of 0.08°C) in 640x480 resolution and encoded using 8 bit per pixel.
Restoration by best and worst results
RMSE - Low noise
NCC low noise
SNR – low noise
RMSE - medium noise
NCC medium noise
PSNR - medium noise level
RMSE - high noise level
NCC - high noise level
PSN - high noise level
Comparison of the average NCC values for all images on all noise level for the used denoising methods.

<table>
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<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
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<td>0.9887</td>
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<td>0.9925</td>
<td>0.9916</td>
<td>0.9910</td>
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</table>
Comparing the measures SNR, NCC and RMSE for each type of wavelet used.
top 50 results

NCC vs Level

Basis

NCC
top 50 results
The 10 best combinations of characteristics for low noise level

<table>
<thead>
<tr>
<th>Base</th>
<th>Level</th>
<th>H/S</th>
<th>NCC</th>
<th>SNR</th>
<th>RMSE</th>
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<td>H</td>
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Results for the best case of each noise level.

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<th>Noise l.</th>
<th>Base</th>
<th>Dec.l.</th>
<th>HVS</th>
<th>Thr. index</th>
<th>SNR</th>
<th>RMSE</th>
<th>NCC</th>
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<td>1.098</td>
<td>0.999</td>
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<td>183</td>
<td>16.850</td>
<td>7.611</td>
<td>0.989</td>
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</tbody>
</table>

Conclusion:

Averaging all noise level:

The most relevant are:
Coiflet 1, Symmlet 2, Daubechie 2, Symmlet 3, Daubechie 3, Biortogonal 2.6 and Reverse biortogonal 5.5.

The hard threshold is always better.

For low level of noise only the three levels of decomposition can be used.
Famous example of Daubechies (1993) denoise

Donoho denoise:
- Coiflets-3
- threshold
- inverse
Second part

- Use these conclusions for the database project.
Steps to perform an efficient restoration scheme for infrared images considering the noise level.

1: Image acquisition and storage as a raw data

2: Evaluation of noise level and decision about decomposition in level 3 or 4.

3: Coiflet wavelet and hard threshold are used.

4: Coefficients for thresholding is select automatically based on the NCC.

5: The image is reconstructed using the modified coefficients.
Restoration of real infrared of whatever noise levels
Comparing achieved characteristics for typical breast image.

Results

Original : 49.519 bytes (1),
image after storage and transmission:
50.846 byte (2) and
denoised image by the proposed scheme:
15.869 bytes (3).

<table>
<thead>
<tr>
<th>images</th>
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<th>RMSE</th>
<th>NCC</th>
<th>Size (bytes)</th>
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<td>15.869</td>
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</table>
Thanks

- [aconci@ic.uff.br](mailto:aconci@ic.uff.br)
- [www.ic.uff.br/~aconci](http://www.ic.uff.br/~aconci)
- [http://200.20.11.171/proeng/](http://200.20.11.171/proeng/)
PROENG’s WebSite

Processamento e Análise de Imagens Aplicadas à Mastologia

Os novos equipamentos digitais para aquisição de imagens médicas (câmeras termográficas, ultrasom, mamogramas digitais, etc) permitem que se pense em combinar as informações anatômicas das diversas fontes para as especificidades dos pacientes. Estes dados (imagens) devem ser processados para realizar e extrair características. Nesta área, técnicas de processamento de imagens e reconhecimento de padrões são muito importantes, tanto para automatizar certos

http://200.20.11.171/proeng
Transformada de Wavelet Contínua

Transformada de Wavelet Contínua é a integral ao longo do tempo de um sinal multiplicado por uma escala, e deslocado por uma função wavelet \((\Psi)\), também chamada wavelet mãe:

\[
C(\text{escala , posição}) = \int_{-\infty}^{\infty} f(t) \Psi(\text{escala , posição , } t) \, dt
\]

O número ao lado do nome da wavelet representa o número de momentos nulos!

\[
\Psi_{a,b}(t) = \frac{1}{\sqrt{a}} \Psi\left(\frac{t-b}{a}\right), \quad a \neq 0, \quad b \in \mathbb{R}
\]

\(a\) define a escala e \(b\) a translação
Repare que mais de uma forma (função base ou mãe) pode ser usada para **gerar uma família**

Para ser considerada uma wavelet, uma função inversível tem que:

Ter uma área total NULA sob a curva da função (ou integral nula) ; e

$$\int \Psi (t) \, dt = 0$$

Ter energia (ou integral do quadrado da função) finita,

$$C_\Psi = 2 \pi \int |u|^{-1} \left| \hat{\Psi}(u) \right|^2 \, du < \infty$$
Para que um \( f \) seja uma \( \Psi \)

- Área zero
  \[
  \int_{-\infty}^{\infty} \psi(t) dt = 0
  \]
  
  energia finita:
  \[
  \int_{-\infty}^{\infty} |\psi(t)|^2 dt
  \]
  
  Tem que ter um suporte compacto
  
  - >
  
  o que significa que ela deve desaparecer fora de um intervalo finito
A Transformada de *Wavelets* contínua em $F(a,b)$ é:
(repare que é uma função de dois parâmetros reais, $a$ e $b$)

$$F(a,b) = \int f(t) \Psi_{a,b}(t) \, dt$$

A função $\Psi_{a,b}(t)$ é denominada função *wavelet e definida como*:

$$\Psi_{a,b}(t) = \frac{1}{\sqrt{a}} \Psi\left(\frac{t-b}{a}\right), \quad a \neq 0, \quad b \in \mathbb{R}$$

A transformada de wavelet decompõe uma função definida no domínio do tempo em outra função, definida no domínio do tempo e no domínio da frequência.
Wavelet Transform

a -> scala

\[ f(t) = \Psi(t); \ a = 1 \]

\[ f(t) = \Psi(2t); \ a = 1/2 \]

\[ f(t) = \Psi(4t); \ a = 1/4 \]
Wavelet Transform

b \rightarrow \text{Position}

Wavelet

Same Function: new location
Wavelet Transform

scale, position & time:

mother wavelet $a=1, b=0$

scale

Large coefficients

Small coefficients
Wavelets

\[ F(a,b) = \int f(t) \Psi_{a,b}(t) \, dt \]

\[ \Psi_{a,b}(t) = \frac{1}{\sqrt{a}} \Psi\left(\frac{t-b}{a}\right), \quad a \neq 0, \quad b \in \mathbb{R} \]

\[ C_\Psi = 2 \pi \int |u|^{-1} |\hat{\Psi}(u)|^2 \, du < \infty \]

\[ \hat{\Psi}(0) = 0 \]
A Transformada de Wavelets contínua em $F(a,b)$ é:

$$F(a,b) = \int f(t) \Psi_{a,b}(t) \, dt$$

A função $\Psi_{a,b}(t)$ é denominada wavelet:

$$\Psi_{a,b}(t) = \frac{1}{\sqrt{a}} \Psi\left(\frac{t-b}{a}\right), \quad a \neq 0, \quad b \in \mathbb{R}$$

As funções wavelets devem ter área zero e energia finita:

$$C_{\Psi} = 2\pi \int |u|^{-1} \left| \hat{\Psi}(u) \right|^2 \, du < \infty$$

condição de admissibilidade
Discrete WT

\[ \Psi_{a,b}(t) = \frac{1}{\sqrt{a}} \Psi\left( \frac{t-b}{a} \right), \quad a = 2^j, b = k \cdot 2^j, \quad (j,k) \in \mathbb{Z}^2 \]

*mother wavelet* \( a=1, b=0 \implies j=0 \text{ e } k=0 \)
Haar

- base Haar

\[
\psi(x) = \begin{cases} 
1 & 0 \leq x < \frac{1}{2} \\
-1 & \frac{1}{2} < x \leq 1 \\
0 & \text{otherwise} 
\end{cases}
\]

\[
0 \leq k \leq 2^j - 1
\]

\[
\psi_{j,k}(x) \equiv \psi\left(2^j x - k\right)
\]

mother wavelet \( j=k=0 \)
discrete Haar:
Alfred Haar – 1909
Particular case of Daubechies:

\[ \psi(x) = \begin{cases} 
1 & 0 \leq x < \frac{1}{2} \\
-1 & \frac{1}{2} \leq x \leq 1 \\
0 & \text{otherwise}
\end{cases} \]
base for Haar - > square waves.

Coefficients $c$

$$f(x) = c_0 + \sum_{j=0}^{\infty} \sum_{k=0}^{2^j-1} c_{jk} \psi_{jk}(x).$$

http://mathworld.wolfram.com/HaarFunction.html
EXEMPLO: signal coefficients

\[ f = \begin{pmatrix} 9 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 4 \begin{pmatrix} 1 \\ -1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \]

DC component - average

For 1 wave.

For 2 wavelets

<table>
<thead>
<tr>
<th>médias</th>
<th>Res.</th>
<th>Deta-lhes</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td></td>
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<td>2</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>
Exemplo: 9 7 3 5
Using average and details:

\[ I(x) = c_0^0 \phi_0^0(x) + c_2^0 \phi_2^0(x) + d_0^2 \psi_0^2(x) + d_1^2 \psi_1^2(x) = 8 \times \]

\[ + 4 \times \]

\[ + 1 \times \]

\[ -1 \times \]

\[ I(x) = c_0^0 \phi_0^0(x) + d_0^0 \psi_0^0(x) + d_0^1 \psi_0^1(x) + d_1^1 \psi_1^1(x) = 6 \times \]

\[ + 2 \times \]

\[ + 1 \times \]

\[ -1 \times \]

(this is multi resolution)
Lossless compression

(11 bits x 7 bits)
Codification & reconstruction
Normalized filters

\[ \phi_j(x) := 2^{j/2} \phi(2^j x - i) \]
\[ \psi_j(x) := 2^{j/2} \psi(2^j x - i) \]

- Normalized coefficients

\[ \begin{bmatrix} 6 & 2 & 1 & -1 \\ 6 & 2 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \]

- New filters

\[ l = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1+\sqrt{3}}{4\sqrt{2}} & \frac{3+\sqrt{3}}{4\sqrt{2}} & \frac{3-\sqrt{3}}{4\sqrt{2}} & \frac{1-\sqrt{3}}{4\sqrt{2}} \end{bmatrix} \]
\[ h = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1-\sqrt{3}}{4\sqrt{2}} & -\frac{3-\sqrt{3}}{4\sqrt{2}} & \frac{3+\sqrt{3}}{4\sqrt{2}} & -\frac{1+\sqrt{3}}{4\sqrt{2}} \end{bmatrix} \]
Haar 2D standard decomposition

• using
  \[ + = +1, \]
  \[ - = -1, \]
  \[ \square = 0 \]
Haar 2D piramidal decomposition

\[ - + = +1, \]
\[ - - = -1 , e \]
\[ e = 0 \]
Thresholding

- Percentages of coefficients.
- with only 5% -> almost perfect reconstruction

using Daub4
Wavelets and noise

- wavelet shrinkage or thresholding
performance

- Compression ratio

- Quality:
  - Root Mean Square Error (RMSE),
  - Sign Noise Ratio (SNR) and
  - Peak Sign Noise Ratio (PSNR)
coeficientes de detalhes
Onde está o ruido, na região suave ou nos detalhes?