Proof Search in Nested Sequent Calculi

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WoLLI
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Sequent systems and modal logics

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E.g.:

$$\frac{\Gamma \vdash A}{\Box\Gamma \vdash \Box A}\ \mathsf{k} \qquad\qquad \frac{\Box\Gamma \vdash A}{\Box\Gamma \vdash \Box A}\ 4$$

Solutions: structures with sequents in them

The solution according to internal approaches:

Extend the sequent structure!

By now, there are many ways to do so:

- Higher-level sequents : Sequents of sequents of sequents of...
  [Došen:'85]
- 2-sequents: Streams of sequents
  [Masini:'92]
- Display calculi: structural connectives for all operators
  [Belnap:'82, Wansing:'94, Kracht:'96]
- Nested sequents: Trees of sequents
  [Kashima:'94, Brünnler:'06, Poggiolesi:'09]
- ...

The Question

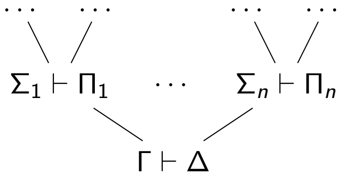
What is the simplest extension of the sequent structure satisfying these desiderata for modal logics?

Case study: Nested sequents

Definition ([Brünnler:'09,Poggiolesi:'09])

A nested sequent is a finite tree whose nodes are labelled with sequents. The interpretation ι of this nested sequent is

$$\bigwedge \Gamma \to \bigvee \Delta \vee \bigvee_{i=1}^{n} \Box \iota(\Sigma_i \vdash \Pi_i) \ .$$



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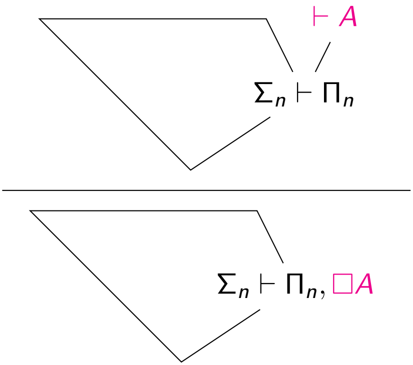

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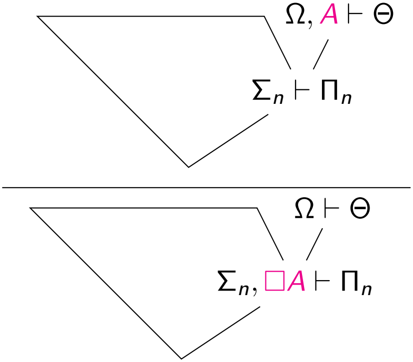

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Trees are nice, but can we go simpler?

Linear nested sequents

Definition

A linear nested sequent is a finite list of sequents, written

$$\Gamma_1 \vdash \Delta_1 /\!\!/ \ \dots /\!\!/ \Gamma_n \vdash \Delta_n$$

and interpreted as $\bigwedge \Gamma_1 \vdash \bigvee \Delta_1 \vee \Box(\dots \Box(\bigwedge \Gamma_n \vdash \bigvee \Delta_n) \dots)$.

The nested sequent system for $\mathtt{K}$ yields the modal rules of $\mathtt{LNS_K}$:

$$\frac{\mathcal{G} /\!\!/ \Gamma \vdash \Delta /\!\!/ \Sigma, A \vdash \Pi /\!\!/ \mathcal{H}}{\mathcal{G} /\!\!/ \Gamma, \Box A \vdash \Delta /\!\!/ \Sigma \vdash \Pi /\!\!/ \mathcal{H}} \ \Box_L \qquad \frac{\mathcal{G} /\!\!/ \Gamma \vdash \Delta /\!\!/ \ \vdash A}{\mathcal{G} /\!\!/ \Gamma \vdash \Delta, \Box A} \ \Box_R$$

Extensions, e.g. (lifted shamelessly from nested sequent calculi):

$$\frac{\mathcal{G} /\!\!/ \Gamma \vdash \Delta /\!\!/ A \vdash}{\mathcal{G} /\!\!/ \Gamma, \Box A \vdash \Delta} \ \mathsf{d} \qquad \frac{\mathcal{G} /\!\!/ \Gamma \vdash \Delta /\!\!/ \Sigma, \Box A \vdash \Pi /\!\!/ \mathcal{H}}{\mathcal{G} /\!\!/ \Gamma, \Box A \vdash \Delta /\!\!/ \Sigma \vdash \Pi /\!\!/ \mathcal{H}} \ 4$$

Completeness for linear nested sequents

We could show completeness via cut elimination . . . but it's easier!

Observation: The data structure of LNS is the same as that of a history in backwards proof search for a sequent calculus.

So we simply simulate a sequent derivation in the last components: ($\mathcal{G}$ is the history)

$$\dfrac{\dfrac{\Gamma \vdash A}{\Box\Gamma \vdash \Box A}\ \mathsf{k}}{\vdots\ \mathcal{G}} \quad \rightsquigarrow \quad \dfrac{\dfrac{\mathcal{G} \mathbin{/\!\!/} \Box\Gamma \vdash \Box A \mathbin{/\!\!/} \Gamma \vdash A}{\mathcal{G} \mathbin{/\!\!/} \Box\Gamma \vdash \Box A \mathbin{/\!\!/} \ \vdash A}\ \Box_L}{\mathcal{G} \mathbin{/\!\!/} \Box\Gamma \vdash \Box A}\ \Box_R$$

Theorem

The LNS calculi for $\mathtt{K}$ *and extensions with axioms from* $\mathrm{d}, \mathrm{t}, 4$ *or* $\mathrm{d}, 4, (4 \wedge 5)$ *are cut-free complete and modular.*

Corollary: Cut-free completeness of the nested sequent calculi.

Application: intuitionistic logic

The same idea connects Maehara's multi-succedent calculus and Fitting's nested sequent calculus for intuitionistic logic, e.g.:

Maehara:

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash \Delta, A \supset B} \supset_R$$

Fitting (restricted to LNS):

$$\frac{\mathcal{G} /\!\!/ \Gamma \vdash \Delta /\!\!/ \Sigma, A \vdash \Pi /\!\!/ \mathcal{H}}{\mathcal{G} /\!\!/ \Gamma, A \vdash \Delta /\!\!/ \Sigma \vdash \Pi /\!\!/ \mathcal{H}} \text{ Lift}$$

$$\frac{\mathcal{G} /\!\!/ \Gamma \vdash \Delta /\!\!/ A \vdash B}{\mathcal{G} /\!\!/ \Gamma \vdash \Delta, A \supset B} \supset_R$$

Maehara's rule is simulated by Fitting's $\supset_R$ and Lift.
The quantifier rules are similar.

Theorem

The LNS calculus for (full) first-order intuitionistic logic (and hence also Fitting's nested sequent calculus) is cut-free complete.

Simply dependent bimodal logics

The language of *simply dependent bimodal logic* $\mathtt{KT} \oplus_{\subseteq} \mathtt{S4}$ contains two modalities $\Box$ and $\heartsuit$, and the axioms are the $\mathtt{KT}$ axioms for $\Box$ together with the $\mathtt{S4}$ axioms for $\heartsuit$ and the *interaction* axiom $\heartsuit A \supset \Box A$.

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$\heartsuit A \supset \Box A$ $\quad$ $\mathsf{k}_{\Box}$ $\Box(A \supset B) \supset (\Box A \supset \Box B)$ $\quad$ $\mathsf{t}_{\Box}$ $\Box A \supset A$ $\quad$ $\dfrac{\vdash A}{\vdash \Box A}\ \mathsf{nec}_{\Box}$

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$$\frac{\mathcal{G} /\!/^* \Gamma \vdash \Delta /\!/^{\Box} \vdash A}{\mathcal{G} /\!/^* \Gamma \vdash \Delta, \Box A}\ \Box_{R\Box} \qquad \frac{\mathcal{S}\{\Gamma \vdash \Delta /\!/^{\Box} \Sigma, A \vdash \Pi\}}{\mathcal{S}\{\Gamma, \Box A \vdash \Delta /\!/^{\Box} \Sigma \vdash \Pi\}}\ \Box_L$$

$$\frac{\mathcal{S}\{\Gamma \vdash \Delta /\!/^{\Box} \Sigma, \heartsuit A \vdash \Pi\}}{\mathcal{S}\{\Gamma, \heartsuit A \vdash \Delta /\!/^{\Box} \Sigma \vdash \Pi\}}\ \heartsuit_{L\Box} \qquad \frac{\mathcal{G} /\!/^* \Gamma \vdash \Delta /\!/^{\heartsuit} \vdash A}{\mathcal{G} /\!/^* \Gamma \vdash \Delta, \heartsuit A}\ \heartsuit_{R\heartsuit}$$

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$$\frac{\mathcal{G} \,/\!/\, \Gamma \vdash \Delta /\!/^{\mathtt{m}} \vdash B}{\mathcal{G} \,/\!/\, \Gamma \vdash \Box B, \Delta} \; \Box_R^{\mathtt{m}} \qquad \frac{\mathcal{G} \,/\!/\, \Gamma \vdash \Delta \,/\!/\, \Sigma, A \vdash \Pi}{\mathcal{G} \,/\!/\, \Gamma, \Box A \vdash \Delta \,/\!/^{\mathtt{m}} \Sigma \vdash \Pi} \; \Box_L^{\mathtt{m}}$$

$$\frac{\mathcal{G} \,/\!/\, \Gamma \vdash \Delta \,/\!/^{\mathtt{m}} \Sigma, A \vdash \Pi}{\mathcal{G} \,/\!/\, \Gamma, \Box A \vdash \Delta \,/\!/^{\mathtt{m}} \Sigma \vdash \Pi} \; \Box_L^{\mathtt{c}} \qquad \frac{\mathcal{G} \,/\!/\, \Gamma \vdash \Delta /\!/ \vdash B}{\mathcal{G} \,/\!/\, \Gamma \vdash \Box B, \Delta} \; \Box_R^{\mathtt{n}}$$

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$$\frac{\mathcal{G} /\!/ \Gamma \vdash \Delta /\!/^{\mathtt{m}} \Sigma, A \vdash \Pi}{\mathcal{G} /\!/ \Gamma, \Box A \vdash \Delta /\!/^{\mathtt{m}} \Sigma \vdash \Pi}\ \Box_L^{\mathsf{c}} \qquad \frac{\mathcal{G} /\!/ \Gamma \vdash \Delta /\!/ \vdash B}{\mathcal{G} /\!/ \Gamma \vdash \Box B, \Delta}\ \Box_R^{\mathsf{n}}$$

$\mathtt{LNS}_{\mathsf{M}}\ \{ \Box_R^{\mathtt{m}}, \Box_L^{\mathtt{m}} \}$ $\mathtt{LNS}_{\mathsf{MC}}\ \{ \Box_R^{\mathtt{m}}, \Box_L^{\mathtt{m}}, \Box_L^{\mathsf{c}} \}$

$\mathtt{LNS}_{\mathsf{MN}}\ \{ \Box_R^{\mathtt{m}}, \Box_L^{\mathtt{m}}, \Box_R^{\mathsf{n}} \}$ $\mathtt{LNS}_{\mathsf{MCN}}\ \{ \Box_R^{\mathtt{m}}, \Box_L^{\mathtt{m}}, \Box_L^{\mathsf{c}}, \Box_R^{\mathsf{n}} \}$

Non-monotone non-normal modal logics

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$$\frac{\mathcal{G} /\!\!/ \Gamma \vdash \Delta /\!\!/^{\mathsf{e}} (\vdash B; B \vdash)}{\mathcal{G} /\!\!/ \Gamma \vdash \Box B, \Delta} \ \Box_R^{\mathsf{e}}$$

$$\frac{\mathcal{G} /\!\!/ \Gamma \vdash \Delta /\!\!/ \Sigma, A \vdash \Pi \quad \mathcal{G} /\!\!/ \Gamma \vdash \Delta /\!\!/ \Omega \vdash A, \Theta}{\mathcal{G} /\!\!/ \Gamma, \Box A \vdash \Delta /\!\!/^{\mathsf{e}} (\Sigma \vdash \Pi; \Omega \vdash \Theta)} \ \Box_L^{\mathsf{e}}$$

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$\mathtt{LNS}_{\mathsf{E}} \quad \{ \Box_R^{\mathsf{e}}, \Box_L^{\mathsf{e}} \}$ $\qquad$ $\mathtt{LNS}_{\mathsf{EC}} \quad \{ \Box_R^{\mathsf{e}}, \Box_L^{\mathsf{e}}, \Box_L^{\mathsf{ec}} \}$

Labelled line sequent systems

$$
\begin{array}{c}
\Gamma_n \vdash \Delta_n \\
| \\
\cdots \\
| \\
\Gamma_1 \vdash \Delta_1 \\
| \\
\Gamma_0 \vdash \Delta_0
\end{array}
$$

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$$\Gamma_n \vdash \Delta_n$$
$$|$$
$$\dots$$
$$|$$
$$\Gamma_1 \vdash \Delta_1$$
$$|$$
$$\Gamma_0 \vdash \Delta_0$$

$$x_{n-1} R x_n, x_0 : \Gamma_0, \dots, x_n : \Gamma_n \vdash x_0 : \Delta_0, \dots, x_n : \Delta_n$$

$$\frac{xRy, X \vdash Y, y : A}{zRx, X, \vdash Y, x : \Box A} \; \Box_R$$

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$$|$$

$$\dots$$

$$|$$

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$$|$$

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$$\frac{xRy, X \vdash Y, y : A}{zRx, X, \vdash Y, x : \Box A} \, \Box_R$$

A LNS calculus is end-active if in all its rules the rightmost components of the premisses are active and the only active components (in premisses and conclusion) are the two rightmost ones.

Focused labelled line sequent systems

$$\frac{zRx : \Gamma; X, x : A, x : B \Rightarrow Y; \Delta}{zRx : \Gamma; X, x : A \wedge B \Rightarrow Y; \Delta} \wedge_L \qquad \frac{zRx : \Gamma; X, x : A \Rightarrow Y, x : B; \Delta}{zRx : \Gamma; X \Rightarrow Y, x : A \supset B; \Delta} \supset_R$$

$$\frac{zRx : \Gamma, x : B_b; X \Rightarrow Y; \Delta}{zRx : \Gamma; X, x : B_b \Rightarrow Y; \Delta} \mathtt{store_L} \qquad \frac{zRx : \Gamma; X \Rightarrow Y; \Delta, x : A_b}{zRx : \Gamma; X \Rightarrow Y, x : A_b; \Delta} \mathtt{store_R}$$

$$\frac{}{zR[x] : \Gamma; X, x : A \rightarrow \cdot; \Delta, x : A} \text{init}$$

$$\frac{zR[x] : \Gamma; X \rightarrow \cdot; \Delta}{zRx : \Gamma; X \Rightarrow \cdot; \Delta} D \qquad \frac{xRy : \cdot; X \Rightarrow Y; \Delta}{[x]Ry : \cdot; X \rightarrow Y; \Delta} R$$

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Focusing effectively blocks derivations where propositional rules are applied between modal ones.

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Focusing effectively blocks derivations where propositional rules are applied between modal ones. Hence we reconcile the added superior expressiveness and modularity of nested sequents with the computational behavior of the standard sequent framework.

Automatic proof search in linear nested sequents

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(init)	$\exists A, x. \lfloor x : A \rfloor^{\perp} \otimes \lceil x : A \rceil^{\perp} \otimes \mathtt{atomic}(A)$
$(\wedge_l)$	$\exists A, B, x. \lfloor x : A \wedge B \rfloor^{\perp} \otimes \lfloor x : A \rfloor \parr \lfloor x : B \rfloor$
$(\wedge_r)$	$\exists A, B, x. \lceil x : A \wedge B \rceil^{\perp} \otimes \lceil x : A \rceil \,\&\, \lceil x : B \rceil$
$(\Box_R)$	$\exists A, B, x. \lceil x : \Box A \rceil^{\perp} \otimes \forall y. (\lceil y : A \rceil \parr R(x, y)) \otimes \exists z. R(z, x)^{\perp}$
$(\Box_L)$	$\exists A, B, x. \lfloor x : \Box A \rfloor^{\perp} \otimes \exists y. (\lfloor y : A \rfloor \parr R(x, y)) \otimes R(x, y)^{\perp}$

Conclusion and future work

Summing up we:

- proposed focused nested sequent systems for a number of modal logics (including a non-trivial bimodal logic and non-normal logics) which match the complexity of existing sequent calculi;

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