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Formulations and heuristics for the multi-item uncapacitated lot-sizing problem with inventory bounds

Rafael A. Melo[a*] and Celso C. Ribeiro[b]

[a] *Departamento de Ciência da Computação, Universidade Federal da Bahia, Salvador, Brazil;* [b] *Instituto de Computação, Universidade Federal Fluminense, Niterói, Brazil*

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We consider the multi-item uncapacitated lot-sizing problem with inventory bounds, in which a production plan for multiple items has to be determined considering that they share a storage capacity. We present (a) a shortest path formulation and (b) a formulation based on the a priori addition of valid inequalities, which are compared with a facility location formulation available in the literature. Two easy-to-implement mixed integer programming heuristic frameworks are also presented, (a) a rounding scheme and (b) a relax-and-fix approach performed in a time partitioning fashion. Computational experiments are performed to evaluate the different approaches. The numerical results show that the proposed relax-and-fix heuristic outperforms all other approaches. Its solutions are within 4.0% of optimality in less than 10 minutes of running time for all tested instances, with mean gaps in the order of 2.1 and 1.8% for instances with more relaxed and tighter capacities, respectively. The obtained solutions were always better than those obtained by a commercial MIP solver running for one hour using any of the available formulations.

Keywords: lot sizing; production planning; mixed integer linear programming; matheuristics; optimisation

1. Introduction

The production planning of goods usually deals with determining production quantities in order to minimise costs and provide customer satisfaction. In many cases, this amounts to providing the goods when they are required. In industrial applications, it is common that the production is limited by the capacity of the storage facilities which, in many situations, can be used to stock different types of items. Therefore, problems with limited storage, i.e. inventory bounds, including those imposed on collections of different items, become very relevant from a practical point of view.

The uncapacitated lot-sizing is a basic production planning problem whose results have been extended very successfully to treat more general problems, see e.g. Akartunali and Miller (2012). Barany, Van Roy, and Wolsey (1984) showed that the (l, S)-inequalities describe the convex hull of solutions for the problem. In addition, extended formulations such as the facility location (Krarup and Bilde 1977) and the shortest path (Eppen and Martin 1987) formulations, which were originally proposed for the uncapacitated lot-sizing, have been adapted to treat several extensions of the problem. An extensive review can be found in Pochet and Wolsey (2006).

Single-item problems with inventory bounds have been studied by several authors. Pochet and Wolsey (1994) gave a complete description of the convex hull of solutions for the single-item uncapacitated lot-sizing with bounds on stocks under Wagner–Whitin costs (Wagner and Whitin 1958). Atamtürk and Küçükyavuz (2005) performed a polyhedral study of an extension of the problem studied in Pochet and Wolsey (1994), with fixed and linear costs on the amount of stock. Wolsey (2006) showed that the variant of the lot-sizing problem with production and delivery time windows with nonspecific orders (i.e. the orders are not client-specific and, therefore, indistinguishable) is equivalent to the single-item lot-sizing with bounds on stocks. Di Summa and Wolsey (2010) gave valid inequalities and extended formulations describing the convex hull of a discrete lot-sizing problem with bounds on the initial stocks. Hwang and van den Heuvel (2012) presented dynamic programming algorithms for the lot-sizing problem with inventory bounds and backlogging. Hwang, van den Heuvel, and Wagelmans (2013) studied dynamic programming algorithms for the lot-sizing with lost sales and bounded inventories.

Multi-item problems with inventory bounds have also been studied, in particular in the context of more general models, see e.g. Park (2005) and Melo and Wolsey (2012). The multi-item extension of the uncapacitated lot-sizing problem with bounds on stocks was studied by Lange (2010) in the context of transportation planning of reusable packages. The authors presented a facility location formulation (Krarup and Bilde 1977) and a family of simple valid inequalities. Akbalik, Penz, and Rapine (2015) further analysed the computational complexity associated with the problem. Gutiérrez et al. (2013) studied

*Corresponding author. Email: melo@dcc.ufba.br

a weighted variant of this problem in which the weights may represent e.g. the storage capacity used by each item. In that problem, bounds are imposed on the total weight of the stock at the start of the period plus the amount produced in that period.

Relax-and-fix is a mixed integer programming (MIP) based heuristic which has been successfully applied to various NP-hard production planning problems. The approach consists in decomposing the problem into more tractable, smaller subproblems, which are sequentially solved using information from previously solved subproblems. The approach was applied in Federgruen and Tzur (1999) to multi-item one-warehouse multi-retailer problems, in Stadtler (2003) to the multi-level lot-sizing with set-up times and multiple constrained resources, in Federgruen, Meissner, and Tzur (2007) to multi-item capacitated lot-sizing problems, in Akartunali and Miller (2009) to big bucket multi-level problems, and to several others. Other MIP-based heuristics can be encountered in Brahimi, Aouam, and Aghezzaf (2015), Eppen and Martin (1987), Katok, Lewis, and Harrison (1998), Melo and Wolsey (2012), Sambasivan and Yahya (2005), and Steinrücke (2015).

In this paper, we study MIP formulations for the multi-item uncapacitated lot-sizing with inventory bounds and show how they can be used to devise an effective MIP heuristic approach.

In Section 2, we describe the multi-item uncapacitated lot-sizing problem with inventory bounds. The formulations are presented and their linear relaxation bounds are compared in Section 3. Two MIP heuristics making use of these formulations are presented in Section 4: the first is a simple rounding-based heuristic following the idea of Eppen and Martin (1987), while the second is a relax-and-fix approach based on the ideas of time partitioning and rolling horizon. Computational experiments comparing the different approaches are reported in Section 5. The numerical results showed that the relax-and-fix heuristic is able to encounter reasonably good solutions in no more than a few minutes, outperforming a standard MIP solver running for one hour. Final remarks are drawn in the last section.

2. Problem description

The multi-item uncapacitated lot-sizing with inventory bounds deals with the single-level production planning of NI items over a finite time horizon of NT periods. The demand for each item $i = 1, \dots, NI$ at each period $t = 1, \dots, NT$ is represented by d_t^i and the total amount of stock in period $t = 1, \dots, NT$ is limited by u_t. There is a fixed set-up cost f_t^i, a variable production cost $\tilde{p}_t^i$ and a storage cost h_t^i associated with each item $i = 1, \dots, NI$ and each period $t = 1, \dots, NT$. We assume that there are no initial and final stocks and that the demands and costs are nonnegative. The demand for a given item in an interval $[k, t]$ is denoted by $d_{kt}^i = \sum_{l=k}^{t} d_l^i$.

For each item $i = 1, \dots, NI$ and each period $t = 1, \dots, NT$, let variables x_t^i be the amount of item i produced in period t; s_t^i be the amount of item i in stock at the end of period t; and $y_t^i = 1$ if item i is produced in period t, $y_t^i = 0$ otherwise. The problem can be cast as the following MIP model:

$$(MI\text{-}ULS\text{-}IB) \qquad z_{STD} = \min \sum_{i=1}^{NI} \sum_{t=1}^{NT} (h_t^i s_t^i + \tilde{p}_t^i x_t^i + f_t^i y_t^i)$$

$$s_{t-1}^i + x_t^i = d_t^i + s_t^i, \quad i = 1, \dots, NI; \ t = 1, \dots, NT, \tag{1}$$

$$x_t^i \le M y_t^i, \quad i = 1, \dots, NI; \ t = 1, \dots, NT, \tag{2}$$

$$\sum_{i=1}^{NI} s_t^i \le u_t, \quad t = 1, \dots, NT, \tag{3}$$

$$x_t^i, s_t^i \ge 0, \quad i = 1, \dots, NI; \ t = 1, \dots, NT, \tag{4}$$

$$y_t^i \in \{0, 1\}, \quad i = 1, \dots, NI; \ t = 1, \dots, NT. \tag{5}$$

The objective function minimises the sum of storage costs, variable production costs and fixed production costs. Equalities (1) are balance constraints. Inequalities (2) are set-up enforcing constraints. Constraints (3) limit the total stock at a given period. Inequalities (4) and (5) are, respectively, nonnegativity and integrality constraints on the variables.

Since $s_t^i = \sum_{k=1}^{t} (x_k^i - d_k^i)$ for any $i = 1, \dots, NI$ and $t = 1, \dots, NT$, the objective function of $(MI\text{-}ULS\text{-}IB)$ can be replaced by

$$\min \sum_{i=1}^{NI} \sum_{t=1}^{NT} (p_t^i x_t^i + f_t^i y_t^i) + \sum_{t=1}^{NT} d_t \left(\sum_{k=t}^{NT} h_k \right),$$

where $p_t^i = \tilde{p}_t^i + \sum_{k=t}^{NT} h_k$. As $\sum_{t=1}^{NT} d_t (\sum_{k=t}^{NT} h_k)$ is a constant, the objective function becomes equivalent to

$$\min \sum_{i=1}^{NI} \sum_{t=1}^{NT} (p_t^i x_t^i + f_t^i y_t^i). \tag{6}$$

Observation 1 below states the fact that feasible solutions for the multi-item uncapacitated lot-sizing problem with inventory bounds are composed of solutions for the single-item uncapacitated lot-sizing problems associated with each item such that the sum of the stocks for all these items does not exceed the inventory bounds.

Observation 1 *The set of feasible solutions to problem* (MI-ULS-IB) *can be described by*

$$X^{MI\text{-}ULS\text{-}IB} = \left(\prod_{i=1}^{NI} X^{LS-U_i}\right) \bigcap \left\{ s : \sum_{i=1}^{NI} s_t^i \le u_t,\ t = 1, \dots, NT \right\},$$

where X^{LS-U_i} *is the set of feasible solutions for the uncapacitated lot-sizing problem* $(LS - U_i)$ *for item* $i = 1, \dots, NI$, *given by*

$$s_{t-1}^i + x_t^i = d_t^i + s_t^i, \quad t = 1, \dots, NT, \tag{7}$$

$$x_t^i \le M y_t^i, \quad t = 1, \dots, NT, \tag{8}$$

$$x_t^i, s_t^i \ge 0, \quad i = 1, \dots, NI;\ t = 1, \dots, NT, \tag{9}$$

$$y_t^i \in \{0, 1\}, \quad i = 1, \dots, NI;\ t = 1, \dots, NT. \ \square \tag{10}$$

3. Reformulations

In this section, we describe reformulations for the multi-item uncapacitated lot-sizing problem with inventory bounds (MI-ULS-IB). The first one is the facility location formulation used in Lange (2010) as an extension of the approach presented in Krarup and Bilde (1977). The second is an extension of the shortest path formulation proposed by Eppen and Martin (1987) to (MI-ULS-IB). The other two reformulations make use of (l, S)-inequalities, proposed by Barany, Van Roy, and Wolsey (1984), in order to strengthen relaxations of the (MI-ULS-IB).

3.1 *Facility location formulation*

The facility location formulation is obtained by treating each demand period $t = 1, \dots, NT$ for each item $i = 1, \dots, NI$ as a single facility that can be served from any production period not later than t. In this formulation, the new variables w_{lt}^i represent the amount of item i that is produced in period l to satisfy the demand of period t. The facility location formulation is

$$(FL) \qquad z_{FL} = \min \sum_{i=1}^{NI} \sum_{t=1}^{NT} (f_t^i y_t^i + p_t^i x_t^i)$$

$$\sum_{l=1}^{t} w_{lt}^i = d_t^i, \quad i = 1, \dots, NI;\ t = 1, \dots, NT, \tag{11}$$

$$w_{lt}^i \le d_t^i y_l^i, \quad i = 1, \dots, NI;\ l = 1, \dots, NT;\ t = l, \dots, NT, \tag{12}$$

$$\sum_{i=1}^{NI} \sum_{l=1}^{t} \sum_{j=t+1}^{NT} w_{lj}^i \le u_t, \quad t = 1, \dots, NT, \tag{13}$$

$$x_t^i = \sum_{l=t}^{NT} w_{tl}^i, \quad i = 1, \dots, NI;\ t = 1, \dots, NT, \tag{14}$$

$$w_{lt}^i \ge 0, \quad i = 1, \dots, NI;\ l = 1, \dots, NT;\ t = l, \dots, NT, \tag{15}$$

$$x_t^i \ge 0, \quad i = 1, \dots, NI;\ t = 1, \dots, NT, \tag{16}$$

$$y_t^i \in \{0, 1\}, \quad i = 1, \dots, NI;\ t = 1, \dots, NT. \tag{17}$$

Constraints (11) imply that the demand for item i in period t is satisfied. Inequalities (12) are set-up enforcing constraints. Constraints (13) limit the total stock at a given period. They consider that the amount of stock of item i at the end of period t is equal to the total production until period t to satisfy demands that occur later than t, given by $s_t^i = \sum_{l=1}^{t} \sum_{j=t+1}^{NT} w_{lj}^i$. Constraints (14) link the facility location variables with the original ones. Inequalities (15) and (16) are nonnegativity constraints, while (17) are integrality constraints.

3.2 *Shortest path formulation*

In order to describe the shortest path formulation, we consider variable ϕ^i_{lt} to be the fraction of d^i_{lt} to be produced in period l:

$$(SP) \qquad z_{SP} = \min \sum_{i=1}^{NI} \sum_{t=1}^{NT} (f^i_t y^i_t + p^i_t x^i_t)$$

$$\sum_{t=1}^{NT} \phi^i_{1t} = 1, \quad i = 1, \dots, NI, \tag{18}$$

$$\sum_{k=1}^{t-1} \phi^i_{k,t-1} - \sum_{k=t}^{NT} \phi^i_{tk} = 0, \quad i = 1, \dots, NI, \ t = 2, \dots, NT \tag{19}$$

$$\sum_{t=1}^{NT} \phi^i_{t,NT} = 1, \quad i = 1, \dots, NI, \tag{20}$$

$$\sum_{l=t}^{NT} \phi^i_{tl} \le y^i_t, \quad i = 1, \dots, NI; \ t = l, \dots, NT, \tag{21}$$

$$\sum_{i=1}^{NI} \sum_{l=1}^{t} \sum_{j=t+1}^{NT} d^i_{tj} \phi^i_{lj} \le u_t, \quad t = 1, \dots, NT, \tag{22}$$

$$x^i_t = \sum_{l=t}^{NT} d^i_{tl} \phi^i_{tl}, \quad i = 1, \dots, NI; \ t = 1, \dots, NT, \tag{23}$$

$$\phi^i_{lt} \ge 0, \quad i = 1, \dots, NI; \ l = 1, \dots, NT; \ t = l, \dots, NT, \tag{24}$$

$$x^i_t \ge 0, \quad i = 1, \dots, NI; \ t = 1, \dots, NT, \tag{25}$$

$$y^i_t \in \{0, 1\}, \quad i = 1, \dots, NI; \ t = 1, \dots, NT. \tag{26}$$

Equalities (18)–(20) are shortest path constraints defining flow conservation requirements related to the ϕ variables for each item. The reader is referred to Pochet and Wolsey (2006) for a more detailed explanation of shortest path formulations applied to lot-sizing problems. Inequalities (21) are set-up enforcing constraints. Constraints (22) limit the total stock at a given period. Constraints (23) link the shortest path variables with the original ones. Inequalities (24) and (25) are nonnegativity constraints, while expressions (26) correspond to integrality constraints.

3.3 *Formulations using (l, S)-inequalities*

The next formulation uses (l, S)-inequalities (Barany, Van Roy, and Wolsey 1984) and, differently from the previous formulations, does not rely on additional variables. It is obtained by adding valid inequalities to the standard formulation in order to achieve improved bounds. Given an interval $[l, t]$ define $L = \{l, \dots, t\}$ and $S \subseteq L$. The formulation is

$$(LS) \qquad z_{LS} = \min \sum_{i=1}^{NI} \sum_{t=1}^{NT} (p^i_t x^i_t + f^i_t y^i_t)$$

$$(1)-(5)$$

$$s^i_{l-1} + \sum_{u \in S} d^i_{ut} y^i_u + \sum_{u \in L/S} x^i_u \ge d^i_{lt}, \quad i = 1, \dots, NI; \ l = 1, \dots, NT; \ t = l, \dots, NT. \tag{27}$$

Constraints (27) are the well-known (l, S)-inequalities for the uncapacitated lot-sizing problem, which imply that the demand d^i_{lt} is either satisfied by the stock available at the end of period $l - 1$ or there should be additional production in periods $l, l + 1, \dots, t$, forcing some set-up variables to take nonzero values. Note, however, that there is an exponential number of such inequalities and their practical use requires the implementation of a cutting-plane algorithm.

Definition 1 The costs are said to be nonspeculative (or Wagner–Whitin) if $\tilde{p}^i_t + h^i_t \ge \tilde{p}^i_{t+1}$, for all $i = 1, \dots, NI$ and for all $t = 1, \dots, NT - 1$ or, alternatively, $p^i_t \ge p^i_{t+1}$ in terms of the modified costs that appear in the objective function (6).

Situations with nonspeculative costs are very frequent in practical situations. We remark that such assumption appears in the instances considered in several recent studies in the lot-sizing literature.

The last formulation is obtained using just a subset of the (l, S)-inequalities, namely the Wagner–Whitin (l, S)-inequalities (Pochet and Wolsey 1994). The Wagner–Whitin-based formulation can be described as

$$(WW) \qquad z_{WW} = \min \sum_{i=1}^{NI} \sum_{t=1}^{NT} (p_t^i x_t^i + f_t^i y_t^i)$$

$$(1)-(5)$$

$$s_{l-1}^i + \sum_{u=l}^{t} d_{ut}^i y_u^i \geq d_{lt}^i, \quad i = 1, \ldots, NI; \ l = 1, \ldots, NT; \ t = l, \ldots, NT. \tag{28}$$

Constraints (28) are the Wagner–Whitin (l, S)-inequalities. However, we note that, contrary to the case of the general (l, S)-inequalities, there is only a polynomial number of Wagner–Whitin (l, S)-inequalities, which makes it easy to add them a priori in the formulation. As it will be noted later in Proposition 2, in some situations adding only this polynomial family of inequalities may lead to linear relaxation bounds that are as strong as those obtained with the complete family of (l, S)-inequalities.

3.4 *Comparison of the linear relaxation bounds*

In this section, we compare the linear relaxation bounds produced by the different relaxations.

PROPOSITION 1 *Let $\underline{z}_{FL}$, $\underline{z}_{SP}$, and $\underline{z}_{LSI}$ be the linear relaxation bounds provided by the facility location, shortest path and (l, S)-inequalities based formulations, respectively. Then, $\underline{z}_{FL} = \underline{z}_{SP} = \underline{z}_{LSI}$.*

Proof Denote by Q^{FL-U_i}, Q^{SP-U_i} and Q^{LSI-U_i} the polyhedron defined by the linear relaxations of the facility location, shortest path and (l, S)-inequalities based formulations of X^{LS-U_i}, respectively. It is well known that $proj_{x,y,s} Q^{FL-U_i} = conv(X^{LS-U_i})$, $proj_{x,y,s} Q^{SP-U_i} = conv(X^{LS-U_i})$, and $Q^{LSI-U_i} = conv(X^{LS-U_i})$. Together with Observation 1, the above implies that the three linear relaxations correspond to the polyhedron $(\bigcap_{i=1}^{NI} conv(X^{LS-U_i})) \bigcap \{s : \sum_{i=1}^{NI} s_t^i \leq u_t \ t = 1, \ldots, NT\}$ and, therefore, $\underline{z}_{FL} = \underline{z}_{SP} = \underline{z}_{LSI}$. □

PROPOSITION 2 *Let $\underline{z}_{WW}$ be the linear relaxation bound provided by the Wagner–Whitin-based formulation. Then, $\underline{z}_{FL} = \underline{z}_{SP} = \underline{z}_{LSI} = \underline{z}_{WW}$ if the costs are nonspeculative (Wagner–Whitin).*

Proof Denote by Q^{WW-U_i} the polyhedron defined by the linear relaxation of the Wagner–Whitin formulation of X^{LS-U_i}. It was already argued in Proposition 1 that $proj_{x,y,s} Q^{FL-U_i} = proj_{x,y,s} Q^{SP-U_i} = Q^{LSI-U_i} = conv(X^{LS-U_i})$. In addition, under the assumption of nonspeculative costs $Q^{WW-U_i} = conv(X^{LS-U_i})$. Therefore, under nonspeculative costs Q^{WW-U_i} also corresponds to the polyhedron $(\bigcap_{i=1}^{NI} conv(X^{LS-U_i})) \bigcap \{s : \sum_{i=1}^{NI} s_t^i \leq u_t \ t = 1, \ldots, NT\}$ and thus $\underline{z}_{FL} = \underline{z}_{SP} = \underline{z}_{LSI} = \underline{z}_{WW}$. □

COROLLARY 1 *Under general costs (i.e. when costs are not assumed to be nonspeculative), $\underline{z}_{FL} = \underline{z}_{SP} = \underline{z}_{LSI} \geq \underline{z}_{WW}$.*

4. MIP heuristics

In this section, we present a rounding and a relax-and-fix heuristic. For the ease of explanation, we describe the heuristics in general MIP terms, and not specifically applied to the problem treated in this paper. Note that variables x and w used in this section are not the same used in the formulations presented in the previous sections, but they are utilised for the sake of consistency with the standard MIP notation in the literature.

4.1 *Rounding heuristic*

The principle of this heuristic consists in rounding the value of some fractional variables in the optimal solution of a strong linear programming relaxation (i.e. one that gives a good approximation of the convex hull of the feasible solutions) and to fix certain variables in order to allow an optimiser to solve a constrained MIP formulation within a reasonable amount of time (or, at least, find a high quality solution with a small gap).

Consider the integer problem $z^* = \min\{cx : x \in X\}$ with a standard formulation $P^0 = \{x \in \mathbb{R}^n : Ax \geq b\}$ such that $X = P^0 \cap \mathbb{Z}^n$. We also assume there is an extended formulation $P^e = \{(x, w) \in \mathbb{R}^n \times \mathbb{R}^q : A'x + D'w \geq d\}$, with $X = proj_x(P^e) \cap \mathbb{Z}^n$ and $z^* = \min\{cx + 0w : (x, w) \in P^e, \ x \in \mathbb{Z}^n\}$, obtained using additional variables w, which is a better approximation to the convex hull of X than P^0 and consequently implies a better linear relaxation bound. For the sake of simplicity of explanation, we assume that all variables x are binary. Note that this approach can be easily extended

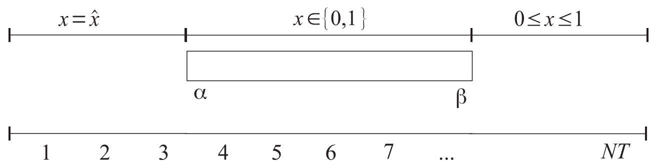

Figure 1. Illustration of the time intervals in the partitioning heuristic.

to more general mixed integer problems by performing all the rounding and fixing procedures only on the integer variables. We now present the heuristic followed by an explanation of its mechanism.

Rounding heuristic
Ad hoc parameters: $\phi^i_{\min}$, $\phi^i_{\max}$ for $i = 1, \dots, n$.

Step 1: Solve the linear program $\min\{cx : (x, w) \in P^e\}$ to obtain a solution $(\hat{x}, \hat{w})$.
Step 2: Determine a neighborhood $N(\hat{x}, \hat{w}) = \{(x, w) \in P^e : x_i = 0 \text{ if } \hat{x}_i \le \phi^i_{\min},\ i = 1, \dots, n\} \cap \{(x, w) \in P^e : x_i = 1 \text{ if } \hat{x}_i \ge \phi^i_{\max},\ i = 1, \dots, n\}$.
Step 3: Solve the integer problem $\min\{cx : (x, w) \in P^e \cap N(\hat{x}, \hat{w}) \cap \{0, 1\}^{n \times q}\}$ to obtain an approximate solution $\bar{x} \in X$.

The algorithm is straightforward and works as follows, assuming that an appropriate extended formulation P^e was already obtained. In Step 1, the linear programming relaxation of the extended formulation is solved. In Step 2, a neighbourhood is defined by fixing each variable x_i according with the solution obtained in Step 1 and with ad hoc parameters $\phi^i_{\min}$ and $\phi^i_{\max}$, for $i = 1, \dots, n$. They are used to determine intervals in which the variables are considered to be close enough to an integer value and, therefore, likely to take its rounded value in a good feasible integer solution. Note that all other variables remain unrestricted. In Step 3, the restricted MIP with some of the variables fixed according with the neighbourhood obtained in Step 2 is solved. This heuristic framework can be easily applied to formulations (*SP*), (*FL*) and (*WW*).

However, we note that this sort of fixations may lead to infeasibilities in some situations and that our framework does not have a backtracking mechanism. For the problem (*MI-ULS-IB*) considered in this work, fixing the binary variables to 0 can lead to infeasibilities using any of the formulations, since they create limitations in the stocks. Fixing variables only to 1, and never to 0, guarantees that all the subproblems are feasible as long as the whole problem also is. We notice that this behaviour can be easily achieved by setting the thresholds $\phi^i_{\min}$ to any negative value, for $1 \le i \le NI$.

4.2 *Relax-and-fix*

We consider an integer problem $z^* = \min\{cx : x \in X\}$ whose variables can be partitioned according to time periods $t \in \{1, \dots, NT\}$. The underlying principle of this heuristic consists in partitioning the time horizon and solving relaxations of some restricted, more manageable subproblems. The algorithm starts with a restrained subproblem in the beginning of the horizon and moves forward towards the end of the time horizon. Once a particular restricted subproblem is solved, some variables are fixed and the algorithm continues to the next subproblem, until the end of the horizon is reached and a feasible solution is obtained.

Using any valid formulation P available to the problem, the heuristic solves a series of consecutive subproblems in which, according to the period to which they correspond, certain integer variables are fixed, some are restrained to be integer and others are relaxed. In our specific case, we consider P to be a partial Wagner–Whitin formulation in which just a subset of the inequalities (28) are added to the formulation, as it will be explained with more details later in this section. Let α and β be integers such that $1 \le \alpha \le \beta \le NT$. Starting from $\alpha = 1$, the algorithm successively solves constrained subproblems while increasing the values of α and β, which define three intervals at each iteration as illustrated in Figure 1. The first interval is composed of all periods $t < \alpha$ and all integer variables associated with periods in this interval are fixed according to the values assumed in one of the previous iterations of the heuristic. The second interval is formed by periods t such that $\alpha \le t \le \beta$ and all integer variables associated with periods in this interval are constrained to take integer values, with its width being $k = \beta - \alpha + 1$. The third interval is formed by all periods $t > \beta$ and all integer variables associated with periods in this interval have their integrality requirements dropped. We now present a general description of the algorithm, followed by an explanation of its working principle.

Table 1. Results comparing the MIP formulations.

Instance	*STD*		*FL*		*SP*		*WW*		WW_{10}	
	bi	gap (%)	*bi*	gap (%)	*bi*	gap (%)	*bi*	gap (%)	*bi*	gap (%)
I_15_50_01	10,122	9.3	9880	5.8	9905	6.1	**9848**	5.5	9947	6.3
I_15_50_02	11,555	9.5	11,207	5.4	**11,150**	4.9	11,415	7.1	11,273	5.8
I_15_50_03	10,230	8.7	**9936**	4.7	9992	5.2	10,027	5.5	10,097	6.0
I_15_50_04	10,641	8.4	**10,210**	3.7	10,347	4.9	10,514	6.4	10,411	5.4
I_15_50_05	9472	8.6	9354	6.3	**9296**	5.7	9524	8.0	9409	6.7
I_15_50_06	12,216	11.2	11,854	6.5	11,742	5.6	11,828	6.2	**11,702**	5.2
I_15_50_07	9807	9.1	9543	5.2	**9423**	4.0	9548	5.2	9551	5.1
I_15_50_08	9110	8.2	8959	5.8	8944	5.6	8938	5.4	**8931**	5.2
I_15_50_09	10,013	8.8	9749	5.4	**9737**	5.2	9799	5.8	9784	5.5
I_15_50_10	10,463	8.5	10,209	5.8	10,189	5.5	**10,162**	5.3	10,278	6.3
I_30_50_01	19,144	9.2	18,502	3.5	18,463	3.3	**18,335**	2.6	18,419	3.0
I_30_50_02	17,215	6.3	16,824	2.6	16,793	2.4	**16,807**	2.5	16,810	2.4
I_30_50_03	20,861	7.4	20,327	3.6	20,206	3.1	20,200	3.0	**20,120**	2.6
I_30_50_04	19,153	8.1	18,632	2.9	**18,587**	2.6	18,646	3.0	18,663	3.1
I_30_50_05	20,271	8.6	**19,502**	2.7	19,539	2.9	19,504	2.7	19,586	3.1
I_30_50_06	19,039	7.3	18,553	2.9	18,670	3.5	**18,492**	2.6	18,638	3.4
I_30_50_07	20,724	7.2	20,342	3.5	**20,239**	3.0	20,309	3.3	20,386	3.7
I_30_50_08	17,430	7.8	16,979	3.2	16,974	3.2	**16,962**	3.1	17,063	3.7
I_30_50_09	18,694	7.6	18,263	3.3	18,251	3.3	**18,177**	2.9	18,251	3.2
I_30_50_10	19,525	9.0	18,807	3.2	18,878	3.5	**18,782**	3.0	18,837	3.2
I_45_50_01	30,064	9.3	**28,614**	1.6	28,747	2.1	28,688	1.9	28,617	1.6
I_45_50_02	29,822	7.5	28,842	1.8	29,027	2.4	28,905	2.0	**28,788**	1.6
I_45_50_03	29,590	8.0	28,631	2.3	28,629	2.3	28,543	2.0	**28,511**	1.9
I_45_50_04	26,807	6.8	26,049	1.9	26,135	2.2	**25,992**	1.7	26,002	1.7
I_45_50_05	32,313	8.9	31,030	2.6	31,113	2.8	30,902	2.2	**30,853**	2.0
I_45_50_06	27,564	7.2	26,745	2.0	26,821	2.3	**26,618**	1.5	26,692	1.8
I_45_50_07	27,853	8.0	**26,849**	1.6	27,002	2.2	26,906	1.8	26,971	2.0
I_45_50_08	31,480	7.8	30,431	2.0	30,615	2.6	30,432	2.0	**30,430**	2.0
I_45_50_09	30,049	7.8	29,260	1.9	29,573	3.0	29,343	2.2	**29,191**	1.7
I_45_50_10	29,102	7.0	**28,271**	1.8	28,537	2.7	28,365	2.1	28,363	2.1
Geometric mean		8.2		3.2		3.4		3.2		3.2
# best solutions	0		6		6		10		8	

Relax-and-fix heuristic
Ad hoc parameters: k, k'.
Initialization: $\alpha = 1$, $\beta = \alpha + k - 1$, deactivate the integrality constraints.

Step 1: Run the MIP over formulation P activating the integrality constraints on all binary variables corresponding to interval $[\alpha, \beta]$.
Step 2: Fix all binary variables in the interval $[\alpha, \alpha + k' - 1]$ to their optimal values in the solution obtained in Step 1.
Step 3: If $\beta < NT$, then set $\alpha \leftarrow \alpha + k'$, $\beta \leftarrow \min\{\alpha + k - 1, NT\}$, and return to Step 1.

In Step 1, a MIP subproblem is solved with the integer variables constrained as described in the explanation of the interval $[\alpha, \beta]$, which in our case corresponds to solving P with inequalities (5) active for $t = \alpha, \ldots, \beta$. In Step 2, all integer variables associated with the interval $[\alpha, \alpha + k' - 1]$ are fixed according to the values they assumed in Step 1, where k' is an ad hoc parameter that determines the width of the fixation horizon, with $1 \leq k' \leq k$. Next, in Step 3 the values of α and β are updated, i.e. increased, and the rolling interval moves toward the end of the horizon. This time partitioning scheme is illustrated in Figure 2. Note that a total of $\lceil NT/k' \rceil$ subproblems have to be solved and that Steps 1–3 are performed until the rolling interval reaches the end of the time horizon (i.e. $\beta = NT$) with an integer feasible solution. We remark that, considering the multi-item uncapacitated lot-sizing problem with inventory bounds, the subproblem associated with each interval (which

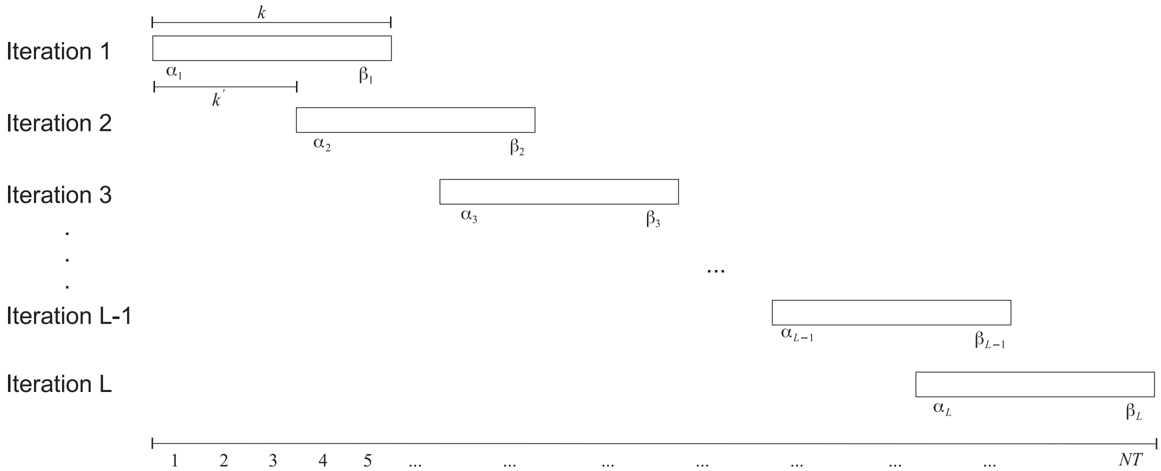

Figure 2. Rolling horizon representation with $k = 5$ and $k' = 3$.

can be solved to optimality or until a small gap is reached within a reasonable amount of time) is NP-hard, and therefore this approach does not lead to a polynomial time heuristic. We also notice that the subproblem obtained by fixing certain variables at some iteration (differently from what happens with problems with capacitated production) is always feasible at the next one due to the fact that there is only capacity on storage and not directly on production. A simple feasible solution to any subproblem is one in which the demands of each period are produced in that period.

We remark that a sequence of MIPs has to be solved in order for the heuristic to terminate, implying that the used formulation P should be effectively solved to optimality or, at least to a small gap, within a small amount of time. For this reason, we consider P as a partial formulation using (WW) in which the integrality constraints of the heuristic are determined by (5). In some situations, large formulations can be used more effectively using the concept of partial formulations (see Van Vyve and Wolsey (2006)), which consists in determining a valid formulation to the problem together with a parameter K limiting its size. In our partial (WW) formulation, inequalities (28) are added only for intervals of size at most $K = 10$, meaning that the inequalities are only inserted into the formulation for pairs (l, t) with $t - l \leq K - 1$.

5. Computational experiments

In this section, we report on computational experiments comparing the different formulations and the proposed heuristics. All experiments were performed on a machine running under Xubuntu x86_64 GNU/Linux, with an Intel Core i7-4770S 3.10 GHz processor, 8 GB RAM memory using FICO Xpress 7.9. All the formulations and heuristics were implemented using Mosel and the solver's default settings were used, with the exception of the optimality tolerance which was set to 10^{-6}.

In order to assess the performance of each approach, we replicated the same type of instance used by Lange (2010), where the author only briefly explored the computational experiments. We also considered instances of larger sizes and with different capacities. In our test set, the number of items $NI \in \{15, 30, 45\}$, the number of periods is $NT = 50$ and the bounds on stocks are set to $u_t = 500$ (for the instances with $NI = 15$ items), $u_t = 1000$ (for the instances with $NI = 30$ items), and $u_t = 1500$ (for the instances with $NI = 45$ items), for $t = 1, \ldots, NT$. An additional instance set was generated with bounds $u_t = 375$ (for the instances with $NI = 15$ items), $u_t = 750$ (for the instances with $NI = 30$ items), and $u_t = 1125$ (for the instances with $NI = 45$ items), for $t = 1, \ldots, NT$. The demands d_t^i are integer values in the interval [0, 25], while the fixed costs f_t^i are time-independent and assume integer values in the interval [20, 150]. There are neither production costs, which are equivalent to the presence of time-invariant production costs as all the demands have to be satisfied, nor storage costs. Ten instances were generated with the previous characteristics for each combination of problem size and capacity configuration.

5.1 *Formulations*

We summarise in Tables 1 and 2 the results obtained with the following formulations: standard (*STD*), facility location (*FL*), shortest path (*SP*), Wagner–Whitin (WW) and Wagner–Whitin adding only the inequalities (28) associated with intervals $[l, t]$ with $t - l < 10$ (WW_{10}). We remark that other values for this parameter could also be chosen, but intervals of size 10 already offer a good trade-off between formulation size and the linear relaxation bound. The first column of the table

Table 2. Results comparing the MIP formulations on the instances with tighter capacities.

	STD		*FL*		*SP*		*WW*		WW_{10}	
Instance	*bi*	gap (%)	*bi*	gap (%)	*bi*	gap (%)	*bi*	gap (%)	*bi*	gap (%)
I_15_50_01B	14,019	8.6	13,686	5.9	13,805	6.7	13,777	6.5	**13,680**	5.7
I_15_50_02B	10,819	6.5	10,693	4.9	10,684	4.9	10,660	4.6	**10,585**	3.9
I_15_50_03B	10,572	7.5	**10,254**	4.0	10,290	4.3	10,362	5.0	10,297	4.2
I_15_50_04B	12,623	7.2	**12,368**	4.7	12,540	6.0	12,561	6.1	12,517	5.7
I_15_50_05B	11,006	6.9	**10,743**	4.2	10,812	4.8	10,912	5.6	10,859	5.1
I_15_50_06B	9715	7.6	9511	5.2	**9437**	4.4	9529	5.3	9531	5.2
I_15_50_07B	12,580	8.1	12,120	4.2	12,200	4.8	12,204	4.8	**12,111**	4.0
I_15_50_08B	15,012	7.3	**14,726**	4.9	14,832	5.6	14,956	6.4	14,802	5.3
I_15_50_09B	14,398	7.8	14,179	5.5	14,182	5.6	14,337	6.5	**14,162**	5.3
I_15_50_10B	13,619	7.2	13,404	5.4	13,341	4.8	13,393	5.3	**13,330**	4.7
I_30_50_01B	25,222	5.4	24,719	2.9	24,727	2.9	**24,544**	2.2	24,614	2.4
I_30_50_02B	22,604	5.4	22,318	3.1	**22,115**	2.2	22,171	2.4	22,134	2.3
I_30_50_03B	26,813	5.1	26,425	2.8	**26,378**	2.6	26,454	2.9	26,460	2.9
I_30_50_04B	24,328	5.3	**23,900**	2.8	24,045	3.4	23,979	3.1	24,043	3.4
I_30_50_05B	21,834	4.4	21,604	2.6	21,701	3.1	**21,567**	2.5	21,616	2.7
I_30_50_06B	26,396	6.2	25,696	2.9	25,675	2.8	**25,507**	2.1	25,658	2.7
I_30_50_07B	29,241	7.6	**28,222**	3.2	28,302	3.5	28,257	3.3	28,247	3.3
I_30_50_08B	24,401	6.7	23,713	3.0	23,736	3.1	**23,678**	2.8	23,786	3.3
I_30_50_09B	23,944	5.8	**23,388**	2.5	23,453	2.8	23,461	2.8	23,485	2.9
I_30_50_10B	21,602	5.1	**21,207**	2.4	21,254	2.6	21,251	2.6	21,397	3.3
I_45_50_01B	36,387	6.3	35,302	2.0	35,433	2.4	35,321	2.1	**35,264**	1.9
I_45_50_02B	40,185	6.5	**38,696**	1.9	38,878	2.4	38,736	2.0	38,737	2.0
I_45_50_03B	33,756	4.7	**32,941**	1.6	33,049	1.9	32,973	1.6	33,024	1.8
I_45_50_04B	37,577	6.1	36,369	2.0	36,344	1.9	36,297	1.8	**36,176**	1.5
I_45_50_05B	33,243	6.1	32,193	1.7	32,243	1.9	**32,068**	1.3	32,154	1.6
I_45_50_06B	35,085	4.7	34,481	1.9	34,650	2.3	34,419	1.7	**34,364**	1.5
I_45_50_07B	35,163	4.7	34,468	1.9	34,586	2.3	34,475	2.0	**34,238**	1.3
I_45_50_08B	40,611	5.9	**39,381**	1.7	39,660	2.4	39,408	1.8	39,394	1.8
I_45_50_09B	34,040	6.7	32,755	1.8	32,926	2.4	32,791	2.0	**32,723**	1.7
I_45_50_10B	37,622	6.0	**36,263**	1.6	36,374	1.9	36,426	2.1	36,348	1.8
Geometric mean		6.2		2.9		3.2		3.0		2.9
# best solutions	0		12		3		5		10	

identifies each instance. For each of the formulations, we give the best integer solution value (*bi*), and the gap at the end of the execution time, calculated as $100 \times (bi - bb)/bi$, with *bb* being the best achieved bound using the corresponding formulation. The solver was run with a time limit of 3600 s for each instance using each of the formulations. Values in boldface indicate the best solution values obtained by the different approaches. Note that we do not report on the time to solve the instances as none of the considered instances could be solved to optimality within 3600 s using any of the formulations.

The results in these tables show, as it was expected, that the solver did not perform well using the standard formulation: there are mean (geometric) open gaps of 8.2 and 6.2% after one hour of running time for instances with more relaxed and tighter capacities, respectively. It can be observed in these tables that stronger formulations performed much better than the standard formulation.

Table 1 shows that *FL*, *WW* and WW_{10} obtained similar mean open gaps (3.2%), with *WW* obtaining the best solution for more instances (10) than the others. Note that *SP* encountered 4 best solutions for instances with 15 items, *WW* obtained 6 best solutions for those with 30 items and WW_{10} found 5 best solutions for the largest instances with 45 items.

The results in Table 2 show that, for instances with tighter capacities, *FL* and WW_{10} obtained lower mean open gaps (2.9%), with *FL* obtaining the best solution for more instances (12) than the others. Observe that WW_{10} encountered the largest number of best solutions for instances with 15 items (five) and 45 items (five), while *FL* and *WW* obtained the largest number of best solutions (four) for instances with 30 items.

Table 3. Results using the rounding heuristic (time limit on each run: 600 s).

	$\phi^i_{\max} = 0.50$			$\phi^i_{\max} = 0.55$			$\phi^i_{\max} = 0.60$			
Instance	*bi*	gap (%)	time (s)	*bi*	gap (%)	time (s)	*bi*	gap (%)	time (s)	bi_{MIP}
I_15_50_01	9855	5.4	106	**9740**	4.3	600	9766	4.6	600	9848
I_15_50_02	11,286	5.9	278	11,246	5.6	600	11,183	5.0	600	**11,150**
I_15_50_03	10,054	5.6	200	10,029	5.4	600	10137	6.4	600	**9936**
I_15_50_04	10,350	4.8	6	10,319	4.6	176	10,303	4.4	600	**10,210**
I_15_50_05	9453	7.1	279	9346	6.1	550	**9260**	5.2	600	9296
I_15_50_06	11,804	6.0	600	11,784	5.8	600	**11,725**	5.3	600	11,742
I_15_50_07	9528	4.8	10	9528	4.8	33	9527	4.8	259	**9423**
I_15_50_08	9034	6.3	330	**8917**	5.1	600	8961	5.6	600	8938
I_15_50_09	9731	5.0	600	**9686**	4.5	600	9687	4.5	600	9737
I_15_50_10	10,125	4.9	7	10,105	4.7	41	**10,093**	4.6	600	10,162
I_30_50_01	18,498	3.4	600	18,351	2.7	600	18,356	2.7	600	**18,335**
I_30_50_02	16,836	2.6	600	16,827	2.5	600	16,812	2.5	600	**16,793**
I_30_50_03	20,241	3.2	600	20,125	2.6	600	**20,030**	2.2	600	20,200
I_30_50_04	18,707	3.3	600	18,604	2.7	600	**18,559**	2.5	600	18,587
I_30_50_05	19,642	3.4	65	19,516	2.8	600	19,516	2.8	600	**19,502**
I_30_50_06	18,631	3.3	600	18,513	2.7	600	18,688	3.6	600	**18,492**
I_30_50_07	20,162	2.6	492	**20,106**	2.3	600	20,112	2.4	600	20,239
I_30_50_08	16,980	3.2	600	16,916	2.8	600	**16,869**	2.6	600	16,962
I_30_50_09	18,299	3.5	600	18,208	3.0	600	18,336	3.7	600	**18,177**
I_30_50_10	18,702	2.6	600	**18,695**	2.5	600	**18,695**	2.5	600	18782
I_45_50_01	28,737	2.0	600	28,644	1.7	600	**28,579**	1.5	600	28,614
I_45_50_02	28,935	2.1	600	**28,833**	1.7	600	28,835	1.8	600	28,842
I_45_50_03	28,593	2.2	563	28,535	2.0	600	**28,386**	1.4	600	28,543
I_45_50_04	26,054	1.9	600	26,011	1.7	600	**25,975**	1.6	600	25,992
I_45_50_05	30,887	2.1	600	**30,834**	1.9	600	30,881	2.1	600	30,902
I_45_50_06	26,700	1.8	600	26,671	1.7	600	26,639	1.6	600	**26,618**
I_45_50_07	27,067	2.4	600	26,951	2.0	600	26,925	1.9	600	**26,849**
I_45_50_08	30,649	2.7	600	**30,401**	1.9	600	30,413	1.9	600	30,431
I_45_50_09	29,235	1.8	600	29,211	1.8	600	**29,140**	1.5	600	29,260
I_45_50_10	28,444	2.4	600	28,309	1.9	600	28,318	2.0	600	**28,271**
Geometric mean		3.3			2.9			2.8		
# best solutions	0			8			11			12

5.2 *Heuristics*

The heuristic frameworks were implemented using partial Wagner–Whitin formulations. Namely, WW_{10} was used for the rounding while WW_k was used for the relax-and-fix heuristic, where k equals the width of the intervals in the rolling horizon scheme. The reason for this choice was not only that the partial Wagner–Whitin formulation provides good linear relaxation bounds, but also that preliminary computational results showed that the larger formulations face difficulties to solve a sequence of restricted problems. In addition, one can easily control the size of a partial Wagner–Whitin formulation by simply setting the width of the interval $[l, t]$ for which the inequalities (28) are added to the formulation.

A time limit of 600 s was imposed on each run. Three different ad hoc parameter values were considered for each of the two heuristics.

As noted earlier in Section 4.1, fixing the y variables to zero in the rounding heuristic can lead to infeasibilities. Therefore, only the variables assuming values greater than or equal to a certain threshold in the linear relaxation are fixed to one. The three values used for the ad hoc parameter $\phi^i_{\max}$ were 0.50, 0.55 and 0.60. The value of $\phi^i_{\max}$ is set to -0.01 so that variables are never fixed to zero.

In the case of the relax-and-fix heuristic, three different combinations were considered for the ad hoc parameters: $k = 6$ and $k' = 3$, $k = 7$ and $k' = 4$, and $k = 8$ and $k' = 5$. Since the time limit for each run is 600 s and there are $\lceil NT/k' \rceil$

Table 4. Results using the rounding heuristic on the instances with tighter capacities (time limit on each run: 600 s).

Instance	$\phi^i_{\max} = 0.50$			$\phi^i_{\max} = 0.55$			$\phi^i_{\max} = 0.60$			
	bi	gap (%)	time (s)	*bi*	gap (%)	time (s)	*bi*	gap (%)	time (s)	bi_{MIP}
I_15_50_01B	13,609	5.3	3	13,596	5.2	23	**13,517**	4.6	513	13,686
I_15_50_02B	10,790	5.7	4	10,790	5.7	75	10,725	5.1	293	**10,660**
I_15_50_03B	10,410	5.2	4	10,355	4.7	557	10,387	5.0	600	**10,254**
I_15_50_04B	12,512	5.7	3	12,478	5.4	3	12,420	5.0	600	**12,368**
I_15_50_05B	10,892	5.4	2	10,818	4.7	600	10,753	4.2	600	**10,743**
I_15_50_06B	9605	5.9	60	9536	5.2	600	9557	5.4	600	**9437**
I_15_50_07B	12,332	5.7	12	12,209	4.7	600	12,230	4.9	600	**12,120**
I_15_50_08B	14,821	5.4	2	14,647	4.3	178	**14,598**	4.0	600	14,726
I_15_50_09B	14,148	5.2	1	14,124	5.1	2	**14,088**	4.8	170	14,179
I_15_50_10B	13,400	5.2	8	13,358	4.9	600	**13,333**	4.7	600	13,341
I_30_50_01B	24,611	2.4	6	24,606	2.4	600	24,595	2.4	600	**24,544**
I_30_50_02B	22,172	2.4	600	22,170	2.4	600	**22,105**	2.1	600	22,115
I_30_50_03B	26,416	2.7	25	**26,315**	2.3	600	26,426	2.8	600	26,378
I_30_50_04B	23,942	3.0	3	23,912	2.8	218	**23,868**	2.6	600	23,900
I_30_50_05B	21,671	2.9	600	21,618	2.7	600	21,576	2.5	600	**21,567**
I_30_50_06B	25,784	3.2	11	25,650	2.7	600	25,707	2.9	600	**25,507**
I_30_50_07B	28,169	3.0	37	28,197	3.1	600	**28,117**	2.8	600	28,222
I_30_50_08B	23,647	2.7	125	23,612	2.5	600	**23,556**	2.3	600	23,678
I_30_50_09B	23,610	3.4	184	23,459	2.8	600	23,538	3.1	600	**23,388**
I_30_50_10B	21,382	3.2	13	21,283	2.7	600	**21,205**	2.4	600	21,207
I_45_50_01B	35,204	1.7	600	35,204	1.7	600	**35,173**	1.6	600	35,302
I_45_50_02B	38,519	1.5	600	38,530	1.5	600	**38,515**	1.4	600	38,696
I_45_50_03B	33,024	1.8	61	33,024	1.8	600	32,975	1.6	600	**32,941**
I_45_50_04B	36,393	2.1	600	36,380	2.0	600	**36,213**	1.6	600	36,297
I_45_50_05B	32,203	1.7	600	32,087	1.4	600	32,096	1.4	600	**32,068**
I_45_50_06B	34,472	1.8	65	34,359	1.5	600	**34,335**	1.4	600	34,419
I_45_50_07B	34,413	1.8	600	34,300	1.5	600	**34,235**	1.3	600	34,468
I_45_50_08B	39,480	2.0	600	39,424	1.8	600	**39,357**	1.7	600	39,381
I_45_50_09B	33,036	2.7	600	32,801	2.0	600	**32,697**	1.7	600	32,755
I_45_50_10B	36,300	1.7	600	36,298	1.7	600	36,461	2.1	600	**36,263**
Geometric mean		3.1			2.8			2.7		
# best solutions	0			1			16			13

subproblems to be solved by this heuristic, each subproblem received initially $600/\lceil NT/k' \rceil$ seconds. After each fixing, the remaining available time was equally divided among the subproblems yet to be solved.

We summarise the results obtained with the rounding heuristic in Tables 3 and 4, and those found by the relax-and-fix heuristic in Tables 5 and 6. In each of the tables, the first column identifies the instances. The next columns present, for each value of the parameters, the best feasible solution value *bi*, the relative gap in per cent between the best solution value and the best available bound achieved at the end of the execution using any of the formulations ($\overline{bb}$), calculated as $100 \times (bi - \overline{bb})/bi$, and the time spent in seconds.

The last column in each of these tables gives the best solution values encountered with the previously considered approaches: bi_{MIP} in Tables 3 and 4 is the best integer solution value obtained using any of the tested formulations within the time limit of 3600 s, as already presented in Tables 1 and 2; while $bi_{MIP,round}$ in Tables 5 and 6 is the best integer solution value between bi_{MIP} (obtained with a time limit of 3600 s) and that found by the rounding heuristic (within the time limit of 600 s). We recall that none of the problem instances could be solved to optimality using the formulations tested in Section 5.1.

As before, values in boldface indicate the best solution values obtained using the different approaches.

The results in Table 3 show that the three choices of parameters for the rounding heuristic allowed the solver to obtain solutions which are competitive with the best ones encountered using the exact formulations, allowing an improvement in

Table 5. Results using the relax-and-fix heuristic (time limit on each run: 600 s).

Instance	$k=6,\ k'=3$			$k=7,\ k'=4$			$k=8,\ k'=5$			
	bi	gap (%)	time (s)	*bi*	gap (%)	time (s)	*bi*	gap (%)	time (s)	$bi_{MIP,round}$
I_15_50_01	9643	3.4	55	9637	3.3	69	**9628**	3.2	117	9740
I_15_50_02	11,033	3.7	104	**11,017**	3.6	161	11,034	3.7	254	11,150
I_15_50_03	**9814**	3.3	83	9834	3.5	94	9815	3.3	119	9936
I_15_50_04	10,232	3.7	108	10,233	3.8	149	**10,186**	3.3	208	10,210
I_15_50_05	9137	3.9	65	9125	3.8	95	**9119**	3.7	172	9260
I_15_50_06	**11,457**	3.1	90	11486	3.4	171	11,516	3.6	263	11,725
I_15_50_07	9476	4.3	70	9415	3.7	105	**9384**	3.4	143	9423
I_15_50_08	8832	4.2	78	8758	3.4	79	**8707**	2.8	184	8917
I_15_50_09	9643	4.1	84	**9579**	3.5	81	9599	3.7	161	9686
I_15_50_10	9964	3.4	80	10,009	3.8	87	**9954**	3.3	138	10,093
I_30_50_01	18,337	2.6	366	18,233	2.0	373	**18,145**	1.6	429	18,335
I_30_50_02	16,769	2.2	304	16,762	2.2	341	**16,649**	1.5	420	16,793
I_30_50_03	20,131	2.7	335	**19,949**	1.8	334	19,953	1.8	388	20,030
I_30_50_04	18,527	2.3	346	**18,434**	1.8	326	18,469	2.0	495	18,559
I_30_50_05	19,401	2.2	345	19,377	2.1	343	**19,323**	1.8	378	19,502
I_30_50_06	18,428	2.3	363	18,433	2.3	321	**18,311**	1.7	441	18,492
I_30_50_07	20,102	2.3	364	20,015	1.9	385	**19,933**	1.5	345	20,106
I_30_50_08	16,877	2.6	339	**16,765**	2.0	378	16,783	2.1	439	16,869
I_30_50_09	18,083	2.3	373	18,143	2.7	401	**17,977**	1.8	383	18,177
I_30_50_10	18,651	2.3	358	**18,590**	2.0	338	18,659	2.3	470	18,695
I_45_50_01	28,688	1.9	418	28,677	1.8	466	**28,522**	1.3	526	28,579
I_45_50_02	29,011	2.4	444	28,754	1.5	463	**28,666**	1.2	538	28,833
I_45_50_03	28,500	1.8	448	28,440	1.6	469	28,445	1.7	540	**28,386**
I_45_50_04	26,219	2.5	439	26,134	2.2	471	**25,960**	1.5	521	25,975
I_45_50_05	30,695	1.5	447	30,731	1.6	414	**30,651**	1.4	445	30,834
I_45_50_06	26,808	2.2	415	**26,548**	1.3	447	26,565	1.3	523	26,618
I_45_50_07	27,014	2.2	420	26,861	1.6	480	**26,799**	1.4	507	26,849
I_45_50_08	30,415	1.9	439	30,433	2.0	450	**30,266**	1.4	538	30,401
I_45_50_09	29,365	2.3	459	29,253	1.9	423	29,251	1.9	541	**29,140**
I_45_50_10	28,355	2.1	458	28,166	1.4	464	**28,164**	1.4	498	28,271
Geometric mean		2.6			2.3			2.1		
# best solutions	2			7			19			2

the best known solution for 18 out of the 30 instances. Note that parameter $\phi^i_{\max} = 0.60$ lead to the best results among the tested values and achieved the lowest mean gap (2.8%).

The competitive results of the rounding heuristic can also be observed in Table 4, allowing an improvement in the best known solution for 17 out of the 30 instances. Again, the parameter $\phi^i_{\max} = 0.60$ performed the best, clearly outperforming the other parameter choices for these instances. Note that although this choice leads to smaller neighbourhoods in which solutions can be obtained, i.e. fewer solution choices, it allows a better exploration via branch-and-bound of the restricted problem within the allowed time limit.

Tables 5 and 6 show that the relax-and-fix heuristic outperformed all other approaches when considering the quality of the solutions found within the imposed time limits. It obtained the best solutions for all but two instances with more relaxed capacities, as it can be observed in Table 5. The variant with larger horizon and fixing widths, i.e. $k = 8$ and $k' = 5$, performed the best. Similar results can be observed in Table 6 when considering the instances with tighter capacities, for which variations of the relax-and-fix could improve 29 out of the 30 best known solutions. Once again, the variant with larger horizon and fixing widths performed the best.

Based on the results, we can see that the MIP heuristics found good quality solutions within just a few minutes considering that we are treating large instances of a problem with big bucket-like constraints, which are constraints per period involving

Table 6. Results using the relax-and-fix heuristic on the instances with tighter capacities (time limit on each run: 600 s).

	$k=6,\ k'=3$			$k=7,\ k'=4$			$k=8,\ k'=5$			
Instance	*bi*	gap (%)	time (s)	*bi*	gap (%)	time (s)	*bi*	gap (%)	time (s)	$bi_{MIP,round}$
I_15_50_01B	13,323	3.2	54	13,372	3.6	70	**13,319**	3.2	111	13,517
I_15_50_02B	10,544	3.5	51	10,497	3.1	104	**10,469**	2.8	170	10,660
I_15_50_03B	10,204	3.3	43	10,191	3.2	49	**10,127**	2.6	79	10,254
I_15_50_04B	12,197	3.2	54	12,156	2.9	64	**12,152**	2.9	116	12,368
I_15_50_05B	10,681	3.5	58	10,664	3.4	82	**10,616**	2.9	155	10,743
I_15_50_06B	9395	3.8	48	9354	3.4	69	**9336**	3.2	125	9437
I_15_50_07B	12,019	3.2	46	11,960	2.8	62	**11,929**	2.5	87	12,120
I_15_50_08B	14,513	3.4	52	14,440	2.9	112	**14,422**	2.8	144	14,598
I_15_50_09B	13,829	3.0	56	13,821	3.0	132	**13,814**	2.9	248	14,088
I_15_50_10B	13,100	3.0	69	13,144	3.3	107	**13,075**	2.8	170	13,333
I_30_50_01B	24,458	1.8	283	24,383	1.5	376	**24,325**	1.3	430	24,544
I_30_50_02B	21,985	1.6	250	21,983	1.6	304	**21,967**	1.5	346	22,105
I_30_50_03B	26,073	1.4	252	26,045	1.3	353	**26,035**	1.3	516	26,315
I_30_50_04B	23,652	1.8	300	23,638	1.7	383	**23,628**	1.7	416	23,868
I_30_50_05B	21,385	1.6	286	**21,355**	1.5	438	21,418	1.8	504	21,567
I_30_50_06B	25,424	1.8	280	25,308	1.4	310	**25,292**	1.3	403	25,507
I_30_50_07B	27,842	1.9	340	**27,738**	1.5	424	27,754	1.6	498	28,117
I_30_50_08B	23,478	2.0	295	23,421	1.7	377	**23,402**	1.7	414	23,556
I_30_50_09B	23,189	1.7	304	23,186	1.7	342	**23,109**	1.3	418	23,388
I_30_50_10B	21,062	1.7	261	21,039	1.6	330	**21,011**	1.5	443	21,205
I_45_50_01B	35,049	1.3	427	**35,026**	1.2	448	35,046	1.3	517	35,173
I_45_50_02B	**38,364**	1.1	408	38,387	1.1	400	38,370	1.1	541	38,515
I_45_50_03B	**32,885**	1.4	460	32,887	1.4	416	32,890	1.4	541	32,941
I_45_50_04B	36,176	1.5	405	**36,084**	1.2	437	36,156	1.4	527	36,213
I_45_50_05B	32,083	1.4	389	32,169	1.6	449	32,093	1.4	510	**32,068**
I_45_50_06B	34,339	1.5	412	34,294	1.3	462	**34,271**	1.3	540	34,335
I_45_50_07B	34,187	1.1	396	34,195	1.1	432	**34,110**	0.9	453	34,235
I_45_50_08B	**39,131**	1.1	436	39,148	1.1	467	39,299	1.5	541	39,357
I_45_50_09B	**32,540**	1.2	399	32,718	1.7	430	32,692	1.6	516	32,697
I_45_50_10B	36,117	1.2	425	36,064	1.1	441	**36,060**	1.1	540	36,263
Geometric mean		1.9			1.8			1.8		
# best solutions	4			4			21			1

multiple items. It is worth mentioning that problems with this characteristic are in general difficult to solve to optimality even when the remaining open gap is already small.

We now analyse how the time limit given to the heuristics can affect the quality of the solutions found. We considered two specific instances and evaluated the behaviour of all previously tested variations of the rounding and relax-and-fix heuristics when the time limit increases from 60 to 1200 s by steps of 60 s. Figures 3 and 4 depict the results for instances I_45_50_09 and I_30_50_04B, respectively. We observe that increasing the time limit does not considerably affect the solutions obtained by the rounding heuristic. The best solutions produced by the rounding heuristic are obtained in less than 240 s and do not change thereafter. However, significant improvements are achieved by the variants of the relax-and-fix heuristic when the time limit is increased. It is important to notice that a small increase in the time limit does not necessarily lead to an improvement in the best obtained solution. This behaviour occurs because fixing according to a slightly better solution to a certain subproblem may lead to more difficult subsequent subproblems. In addition, fixing according to a slightly better solution to a certain subproblem is not necessarily better when we take the complete problem into account. However, it is noticeable that, considering these two instances, in the long run, the relax-and-fix heuristic benefits from larger time limits. We observe that the relax-and-fix heuristic significantly improved the quality of the best solutions found when the time limit increased from 600 to 1200 s.

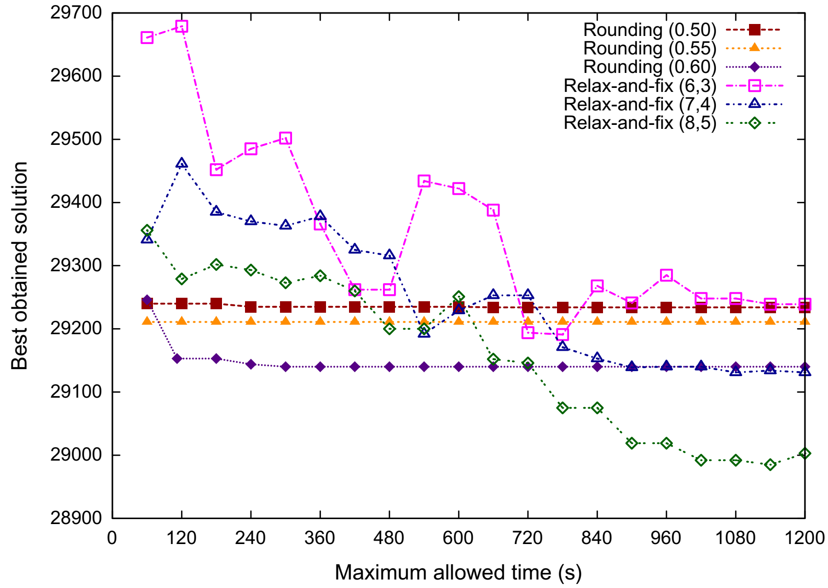

Figure 3. Obtained solutions by the heuristics for instance I_45_50_09 varying the maximum allowed time.

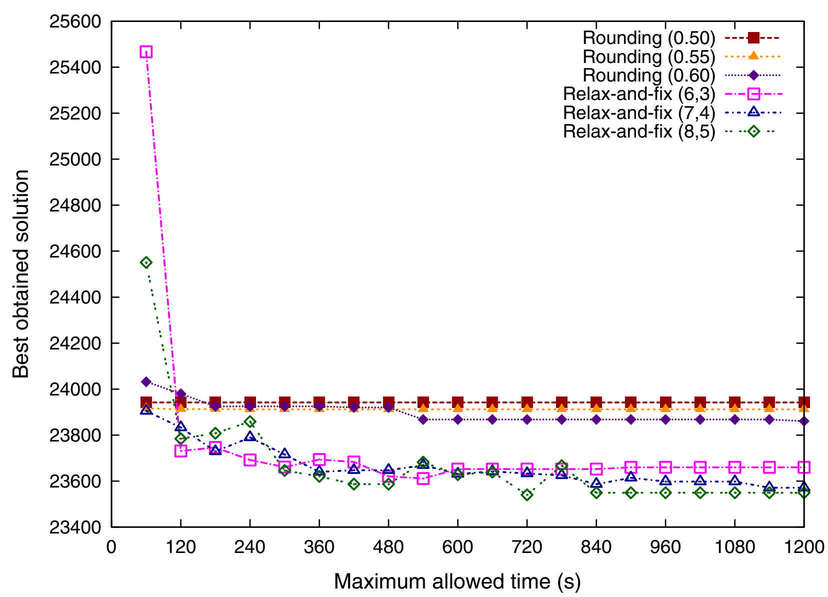

Figure 4. Obtained solutions by the heuristics for instance I_30_50_04B varying the maximum allowed time.

In order to further evaluate the improvements that could be achieved in case some additional running time was allowed to the heuristics, we performed new computational experiments, using the best variant of the relax-and-fix heuristic (with parameters $k = 8$ and $k' = 5$) running for a longer time limit of 1200 s. Table 7 displays, for each instance, the value of the best integer solution found, the remaining open gap and the running time. Values in boldface indicate that the solution found within the time limit of 1200 s is strictly better than that obtained by the same heuristic running within the time limit of 600 s. Values with a superscript 'a' indicate instances in which the solutions obtained within 1200 s were worse than those ones found by the same heuristic running with a time limit of 600 s. Although it is unusual to observe a deterioration in solution quality when the allowed time limit is increased, in certain cases fixing variables according to a slightly better solution to a subproblem may lead to solutions of lesser quality in later subproblems as we noted earlier. These results support

Table 7. Results using the relax-and-fix heuristic with $k = 8$ and $k' = 5$ (time limit on each run: 1200 s).

Instances with more relaxed capacities				Instances with tighter capacities			
Instance	*bi*	gap (%)	time (s)	Instance	*bi*	gap (%)	time (s)
I_15_50_01	9628	3.2	119	I_15_50_01B	13,319	3.2	112
I_15_50_02	**10,993**	3.4	302	I_15_50_02B	10,469	2.8	200
I_15_50_03	9815	3.3	118	I_15_50_03B	10,127	2.6	79
I_15_50_04	[a]10,223	3.7	237	I_15_50_04B	12,152	2.9	122
I_15_50_05	9119	3.7	165	I_15_50_05B	10,616	2.9	168
I_15_50_06	**11,464**	3.2	370	I_15_50_06B	9336	3.2	134
I_15_50_07	9384	3.4	139	I_15_50_07B	11,929	2.5	86
I_15_50_08	[a]8753	3.3	274	I_15_50_08B	14,422	2.8	146
I_15_50_09	9599	3.7	192	I_15_50_09B	13,814	2.9	251
I_15_50_10	9954	3.3	170	I_15_50_10B	13,075	2.8	174
I_30_50_01	**18,121**	1.4	723	I_30_50_01B	[a]24,327	1.3	631
I_30_50_02	[a]16,695	1.8	650	I_30_50_02B	**21,927**	1.3	525
I_30_50_03	**19,897**	1.5	862	I_30_50_03B	**26,024**	1.3	887
I_30_50_04	**18,423**	1.8	774	I_30_50_04B	**23,549**	1.3	807
I_30_50_05	**19,260**	1.5	732	I_30_50_05B	**21,330**	1.4	815
I_30_50_06	[a]18,316	1.7	470	I_30_50_06B	[a]25,309	1.4	785
I_30_50_07	[a]19,935	1.5	645	I_30_50_07B	**27,735**	1.5	684
I_30_50_08	**16,692**	1.5	772	I_30_50_08B	**23,335**	1.4	771
I_30_50_09	**17,931**	1.5	737	I_30_50_09B	**23,086**	1.2	690
I_30_50_10	**18,490**	1.4	702	I_30_50_10B	**20,982**	1.3	620
I_45_50_01	**28,490**	1.2	1028	I_45_50_01B	**34,975**	1.1	956
I_45_50_02	[a]28,704	1.3	1054	I_45_50_02B	**38,322**	0.9	1056
I_45_50_03	**28,249**	1.0	954	I_45_50_03B	**32,696**	0.8	1081
I_45_50_04	**25,829**	1.0	1004	I_45_50_04B	**35,975**	0.9	1024
I_45_50_05	**30,590**	1.2	893	I_45_50_05B	**31,957**	1.0	1023
I_45_50_06	**26,458**	0.9	1033	I_45_50_06B	**34,261**	1.2	935
I_45_50_07	**26,732**	1.2	975	I_45_50_07B	**34,089**	0.8	1041
I_45_50_08	**30,162**	1.1	1066	I_45_50_08B	**39,145**	1.1	929
I_45_50_09	**29,003**	1.1	1081	I_45_50_09B	**32,531**	1.2	1064
I_45_50_10	**28,078**	1.1	1005	I_45_50_10B	**36,058**	1.1	1050
Geometric mean		1.8		Geometric mean		1.6	

the conclusions drawn from Figures 3 and 4. Extending to 1200 s the time limit given to the relax-and-fix heuristic with parameters $k = 8$ and $k' = 5$ leads to improved results for most of the tested instances, with the main exceptions being the smaller instances with 15 items for which most of the subproblems could already be solved to optimality within 600 s.

6. Final remarks

In this paper, we studied the multi-item uncapacitated lot-sizing problem with inventory bounds. We presented (a) a shortest path formulation, (b) a formulation based on the addition of (l, S)-inequalities, (c) a rounding heuristic and (d) a relax-and-fix heuristic based on a rolling horizon time partitioning scheme.

We discussed how these formulations compare in terms of their linear relaxation bounds, also considering an already existing facility location formulation for the problem. The computational experiments have shown that, in general, all tested reformulations performed well even though none of the instances could be solved to optimality. The formulations that performed the best have obtained mean open gaps in the order of 3.2% for the instances with more relaxed capacities (facility location and the Wagner–Whitin-based formulations) and in the order of 2.9% for the instances with tighter capacities (facility location and partial Wagner–Whitin-based formulations), within 3600 s of running time. In terms of the number of best solutions found, the facility location and the partial Wagner–Whitin-based formulations outperformed the others, encountering better solutions than all the others for 18 out of 60 instances.

Different configurations of the proposed heuristics were tested in our computational experiments. The best performing configuration of the rounding heuristic improved the best solution obtained using the formulations for 27 out of the 60 tested instances. In addition, it encountered solutions which are within mean gaps of 2.8 and 2.7% of optimality for the instances with more relaxed and tighter capacities, respectively. The relax-and-fix heuristic clearly outperformed the other approaches in terms of the quality of the solutions found, improving over the best known solutions for all but three of the 60 considered instances. The best parameter configuration of the relax-and-fix heuristic led to solutions within mean optimality gaps in the order of 2.1 and 1.8% for the instances with more relaxed and tighter capacities, respectively.

It was possible to note that, in some cases, the relax-and-fix heuristic may not use all the available time, due to the fact that subproblems solved later in the process tend to be easier than those solved earlier. In consequence, there may be an overestimation of the time reserved for solving them. This observation opens some interesting research directions, such as (a) the study of metrics for estimating the time that should be given to earlier and later subproblems to be solved and (b) the use of the unused time at the end of the execution for additional heuristic procedures, such as local search or fix-and-optimise. The computational experiments also showed that, in general, better solutions could be obtained if additional time is available.

We remark that the approaches proposed in this paper can also be applied to other production planning problems with limited inventory, such as the multi-item dynamic lot-sizing problem with storage capacities considered in Gutiérrez et al. (2013) and the lot-sizing problems with emission constraints studied in Absi et al. (2013) and Retel (2015).

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