Introduction to Geometric Algebra

Lecture 17
Conic Space Model
and Unusual Models of Geometry

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Lecture notes available in
http://www.ic.uff.br/~laffernandes/teaching/2013.1/topicos_ag

Conic Space Model
**Motivation**

Corners, junctions and line segments are important features that can be extracted from edge pixels.

**Conic Space Model**

- Created by Perwass to detect corners, line segments, lines, crossings, y-junctions, and t-junctions as the intersection of conic sections.
The Vector Space of Conics

• It is well known that given a symmetric $3 \times 3$ matrix

\[
A = \begin{pmatrix}
    a_{11} & a_{12} & a_{13} \\
    a_{12} & a_{22} & a_{23} \\
    a_{13} & a_{23} & a_{33}
\end{pmatrix}
\]

the set of vectors $\mathbf{x} = (x \ y \ 1)^T$ that satisfy

$\mathbf{x}^T A \mathbf{x} = 0$

lie on a conic.

The Vector Space of Conics

• This can also be written using the scalar product of matrices, as

\[
X \cdot A = \sum_{i=1}^{3} \sum_{j=1}^{3} x_i a_{ij} = 0
\]

where

\[
X = \mathbf{x} \mathbf{x}^T = \begin{pmatrix}
    x^2 & x y & x \\
    x y & y^2 & y \\
    x & y & 1
\end{pmatrix}
\]
The Vector Space of Conics

• It makes sense to write the same equation using a 6D-vector space of symmetric matrices

\[ x^T \cdot a = x^2 a_{11} + y^2 a_{22} + 2xy a_{12} \\
+ 2xa_{13} + 2ya_{23} + a_{33} = 0 \]

where \( x \) and \( a \) are row matrices encoding, respectively, a point and a conic in 2D base space

Actually, the dual of a conic!

Vectors in the 6D Representational Space

• After some normalization...

\[ x = x e_1 + y e_2 + \frac{1}{\sqrt{2}} e_3 + \frac{x^2}{\sqrt{2}} e_4 + \frac{y^2}{\sqrt{2}} e_5 + xy e_6 \]

Finite Point

\[ a = a_{13} e_1 + a_{23} e_2 + \frac{a_{33}}{\sqrt{2}} e_3 + \frac{a_{11}}{\sqrt{2}} e_4 + \frac{a_{22}}{\sqrt{2}} e_5 + a_{12} e_6 \]

Dual Conic
Replacing Matrix Notation by GA

\[ \mathbf{x} \cdot \mathbf{a} = 0 \]
\[ \mathbf{x} \mathbf{\blacksquare} \mathbf{a} = 0 \]
\[ (\mathbf{x} \mathbf{\blacksquare} \mathbf{a})^{-+} = 0 \]
\[ \mathbf{x} \mathbf{\wedge} (\mathbf{a}^{-+}) = 0 \]
\[ \mathbf{x} \mathbf{\wedge} (\mathbf{a} \mathbf{\blacksquare} \mathbf{I}_6) = 0 \]
\[ \mathbf{x} \mathbf{\wedge} \mathbf{A}_5 = 0 \]

Conics are 5-blades!

Recall that this relations holds for all \( \mathbf{x} \) that lie on the given conic.

Euclidean Metric

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Blades in 2D Conic Space Model

- Let \( \mathbf{x}_i \) be a vector encoding a finite point in 2D conic space model, i.e., (2+4)-D representational space

\[
\begin{align*}
\mathbf{x}_1 & \quad \text{Finite point} \\
\mathbf{x}_1 \mathbf{\wedge} \mathbf{x}_2 & \quad \text{Point pair} \\
\mathbf{x}_1 \mathbf{\wedge} \mathbf{x}_2 \mathbf{\wedge} \mathbf{x}_3 & \quad \text{Point triplet} \\
\mathbf{x}_1 \mathbf{\wedge} \mathbf{x}_2 \mathbf{\wedge} \mathbf{x}_3 \mathbf{\wedge} \mathbf{x}_4 & \quad \text{Point quadruplet} \\
\mathbf{x}_1 \mathbf{\wedge} \mathbf{x}_2 \mathbf{\wedge} \mathbf{x}_3 \mathbf{\wedge} \mathbf{x}_4 \mathbf{\wedge} \mathbf{x}_5 & \quad \text{The conic through the given points}
\end{align*}
\]
**Pseudovectors in 2D Conic Space Model**

- Circle
- Hyperbola
- Ellipse
- Parallel line pair
- Line pair
- Line
- Parabola

**Further Reading**

Unusual Models of Geometry

GA over a space other than a real vector space

• Geometric algebras can be constructed over any type of quadratic space, e.g.,
  ▪ Real-valued vector spaces (we have been working with them)
  ▪ Finite Fourier basis
  ▪ Finite random-variable spaces
  ▪ Basis of orthogonal polynomials
  ▪ Wavelets
  ▪ Spherical harmonics
What is the point?

• The concepts of blades, null spaces, intersections, and combinations of subspaces are still valid.

• They may not have the same geometric meaning.
  - However, a geometric meaning may be given to otherwise abstract operations in this way.

• May help us to gain insights into various fields and to draw parallels between different fields.

Further Reading

• Perwass, C. Geometric Algebra with Applications in Engineering. Springer Publishing Company, 2009. [Chapter 10]