CORS / INFORMS

Montreal - Canada

STRONGER MINIMUM k-TREES RELAXATION FOR VEHICLE ROUTING

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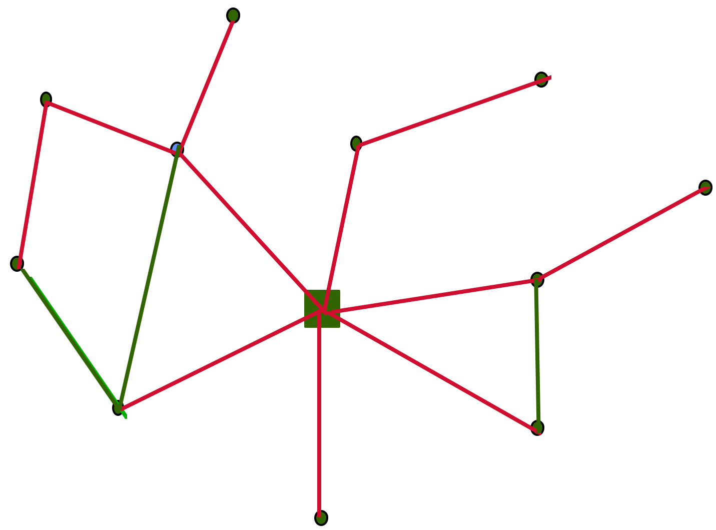
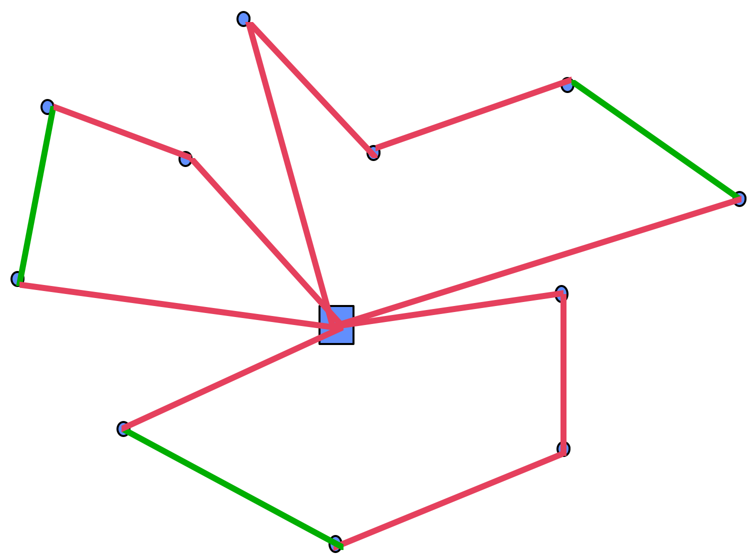
MINIMUM COST K-TREE WITH 2K EDGES INCIDENT ON THE DEPOT (Fisher [94])



3 - TREE (N = 11)



COMPLEXITY : O(n^3) ITERATIONS

FORMULATION

$$z^* = \min \sum_{e \in E} c_e x_e$$

$$s.t. \begin{cases} x(d(i)) = 2; & \forall\, i \in N \\ x(d(S)) \geq 2 \left\lceil d(S) / b \right\rceil; & \forall\, S \subseteq N \;\; with \;\; |S| \geq 2 \end{cases}$$

$$x \in X$$

where:

$$N_0 = N \cup \{0\}$$

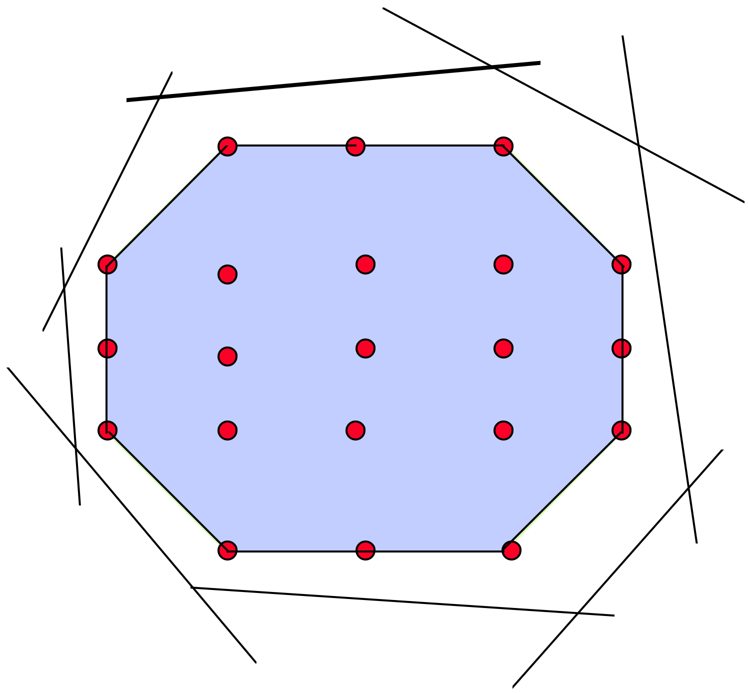
$$S \subseteq N, \quad \overline{S} = N_0 - S$$

$$d(S) = \sum_{i \in S} d_i$$

$$K = \left\lceil \frac{\sum_{i=1}^{n} d_i}{b} \right\rceil$$

X is a K-tree with x(d(0))= 2K

STRONGER FORMULATIONS
(Vehicle Routing)



Capacity constraint (Fisher[94])
Tightened capacity constraint
Combs (Cornuéjols & Harche [93])
Multistars (Araque [91])
....

STRONGER FORMULATIONS

$$\begin{aligned} &\min\ c^T x \\ &s.t\ \ \{ Ax \le b \quad (P1) \\ &\qquad x \in X \end{aligned}$$

$$\begin{aligned} &\min\ c^T x \\ &s.t\ \begin{cases} Ax \le b \\ Bx \le d \end{cases} \qquad (P2) \\ &\qquad x \in X \end{aligned}$$

$$L_0^* = \max_{l \ge 0} \begin{Bmatrix} \min\ c^T x + l^T (Ax - b) \\ s.t \quad x \in X \end{Bmatrix}$$

(DP1)

$$L_k^* = \max_{\substack{l \ge 0 \\ m \ge 0}} \begin{Bmatrix} \min\ c^T x + l^T (Ax - b) + m^T (Bx - d) \\ s.t \quad x \in X \end{Bmatrix}$$

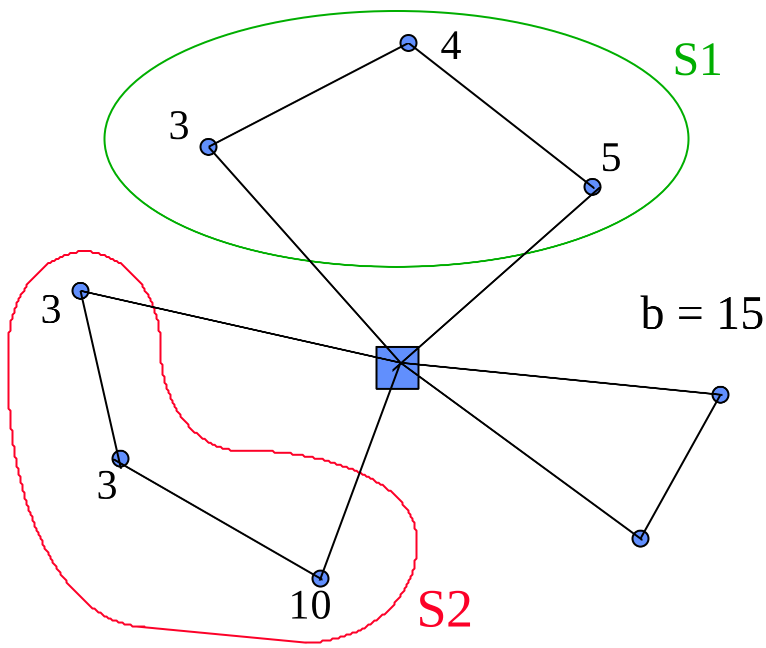
(DP2)

Theorem 1 (Aboudi at al. [91])

Let L_0^* and L_k^*, optimal solutions of dual problems (DP1) and (DP2) respectively. Then $L_0^* \le L_k^*$.

CAPACITY CONSTRAINTS

$$\sum_{i \in S} \sum_{j \in \bar{S}} x_{ij} \geq 2 \left\lceil {d(S)}/{b} \right\rceil; \ \forall S \subseteq N \ \ with \ \ |S| \geq 2$$



$$\sum_{i \in S_1} \sum_{j \in \bar{S}_1} x_{ij} \geq 2 \left\lceil {12}/{15} \right\rceil = 2 \qquad ok!$$

$$\sum_{i \in S_2} \sum_{j \in \bar{S}_2} x_{ij} \geq 2 \left\lceil {16}/{15} \right\rceil = 4 \quad violated!$$

TIGHTENED CAPACITY CONSTRAINTS (Fisher [94])

$$\sum_{j=0}^{n} e_j \sum_{i \in S} x_{ij} \geq 2 \underbrace{\left\lceil \frac{d(S)}{b} \right\rceil}_{\mathrm{r(S)}}, \quad \forall \ S \subseteq N \ \ where \ |S| \geq 2$$

$$e_j = \begin{cases} 0, & j \in S \\ 0, & j \in S' \ \ and \ \ |S'| \leq 2 \\ \dfrac{r(S)}{r(S)+1}, & j \in S' \ \ and \ \ |S'| > 2 \\ 1, & j \in \bar{S} - S' \end{cases}$$

where:

$$S' = \left\{ j \in \bar{S} \,/\, j \geq 1 \ \ and \ \ d_j > br(S) - d(S) \right\}$$

(It is not a facet !!)

IDENTIFICATION OF VIOLATED CAPACITY CONSTRAINTS

$$Let\colon\ D = \left\{ (0, i) \in Tk \,/\, i \in N \right\}$$

Algorithm VCC:

Begin

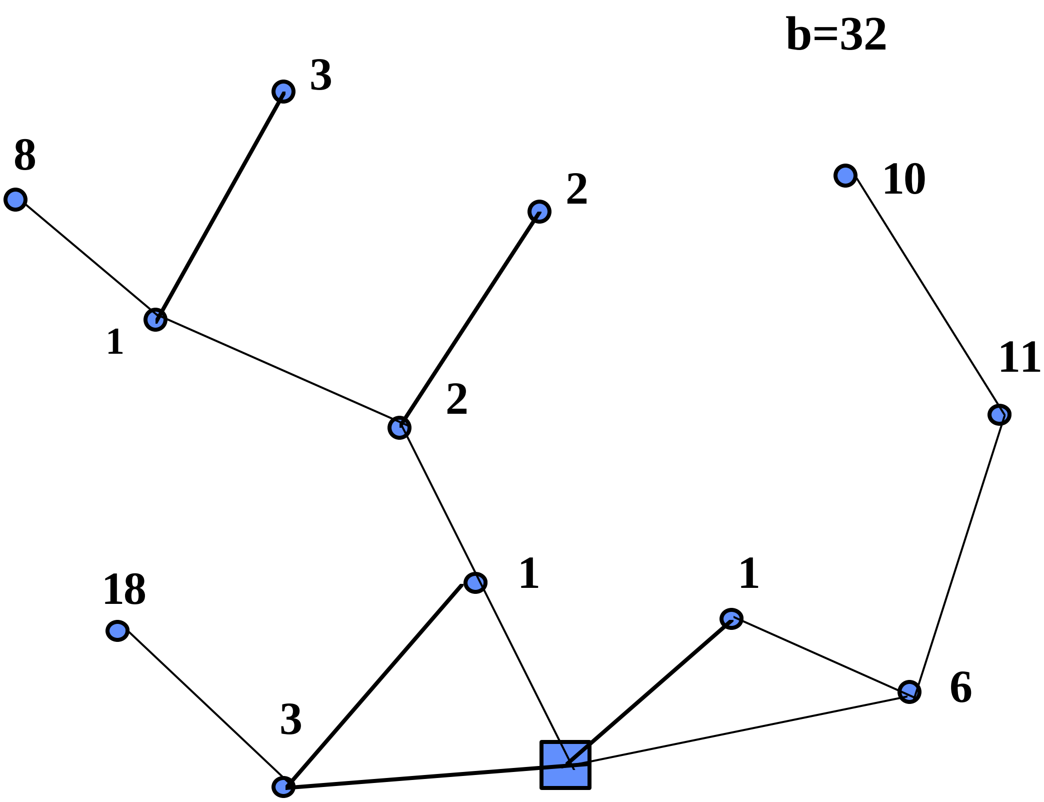
- Construct G'(N, Tk\D);
  (we find m connected components)
- Select sets Si which give us violated sub-tours;

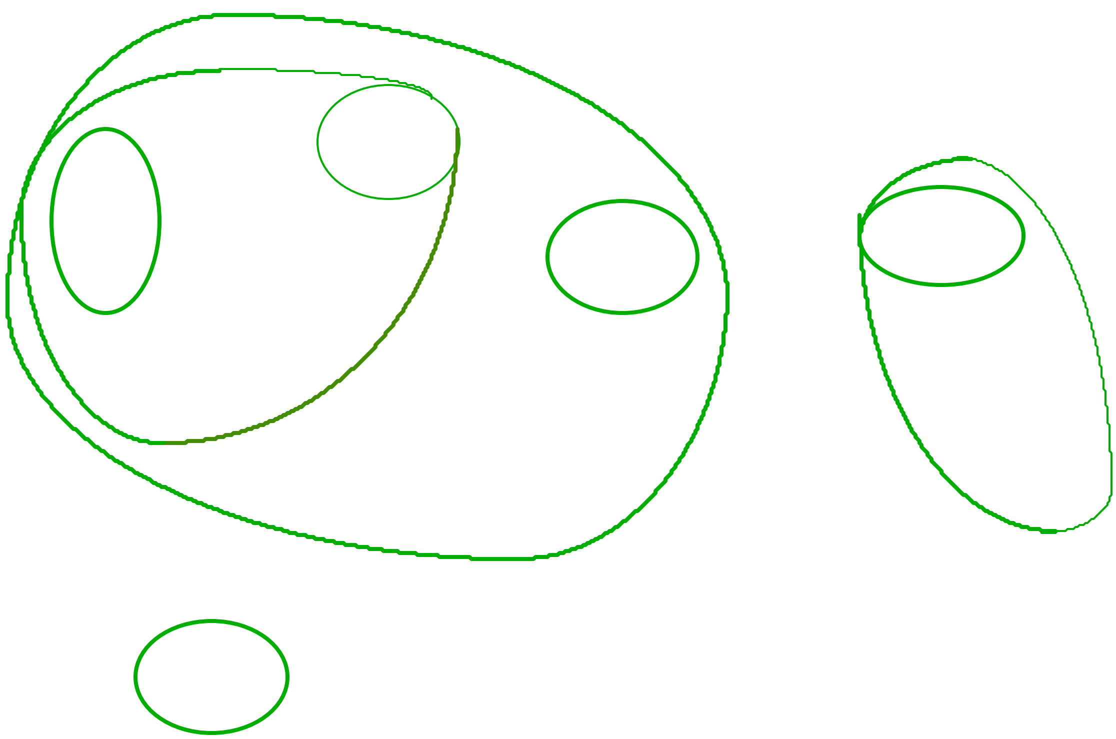
- N(Tk) = { i Î N / x(d(i)) = 1};
- Compute nested sets around each node i Î Tk;

End.

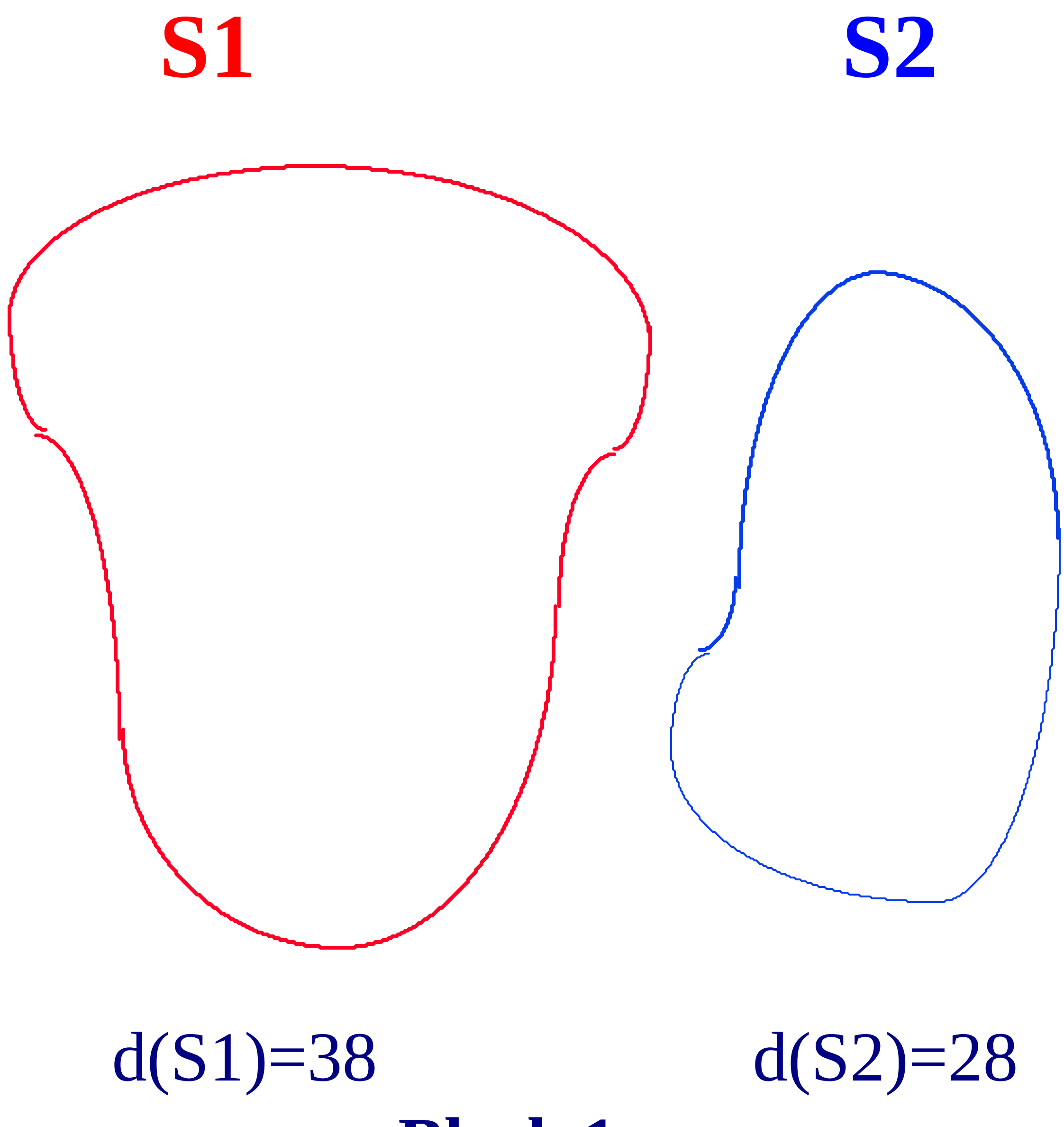
IDENTIFICATION OF VIOLATED CAPACITY CONSTRAINT

2-Tree





Block 2



Block 1

Lemma: Suppose there exist no violated partitions Si at the end of the first block.

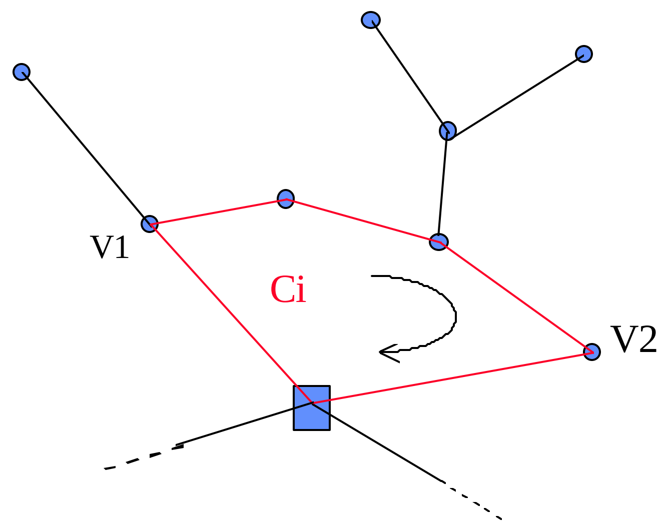
(a) Then $d(S_i) \leq b$ **for i=1,..,m,**

(b) there are exactly 2 edges connecting each component Si to the depot,

(c) each connected component Si defines a tree.

Theorem: Let G(N_0,Tk) be the subgraph generated by a lagrangean problem. The algorithm VCC always find, if there exist one, a violated capacity constraint in Tk.

Proof:



$V_i = S_i \cup \{0\}$ $G_i(V_i, E(V_i))$

COMB INEQUALITIES

(Cornuéjos & Harche [93])

$$x(E(H)) + \sum_{i=1}^{s} x(E(T_i)) \leq |H| + \sum_{i=1}^{s} |T_i| - \frac{3s+1}{2} + a(k-1)$$

where: $H, T_1, \ldots, T_s$ are suth that:

(i) $|T_i \setminus H| \geq 1, \quad for \;\; i = 1, \ldots, s;$

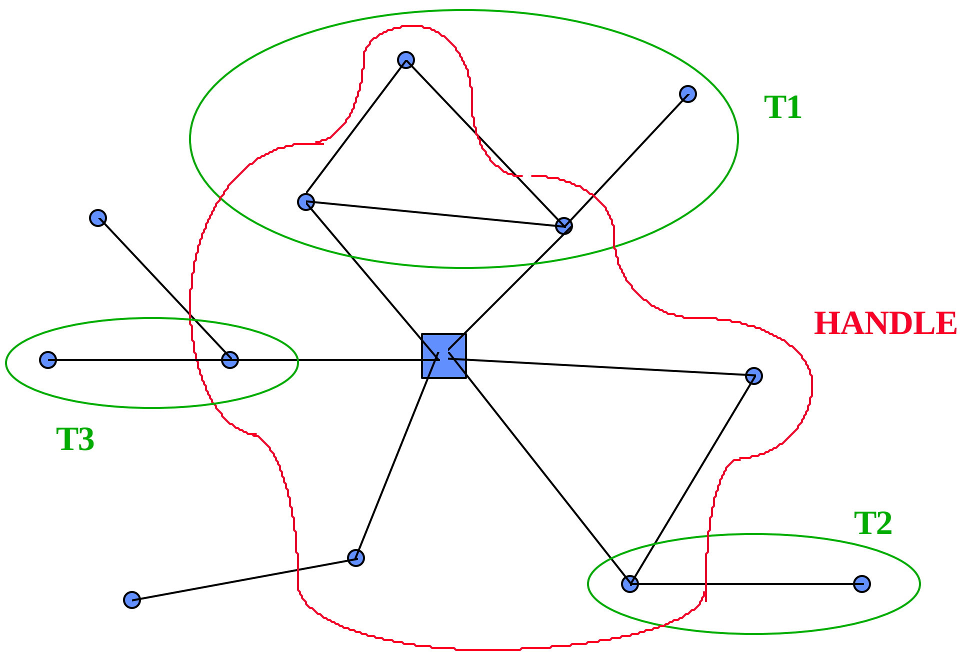
(ii) $|T_i \cap H| \geq 1, \quad for \;\; i = 1, \ldots, s;$

(iii) $|T_i \cap T_j| = 0, \quad for \;\; 1 \leq i \leq j \leq s;$

(iv) s is odd and $s \geq 3$.

$$a = \begin{cases} 0; & if \quad 0 \notin H \cup \left(\bigcup_{i=0}^{s} T_i \right) \\ 1; & if \quad 0 \in H \setminus \left(\bigcup_{i=0}^{s} T_i \right) \\ 2; & if \quad 0 \in H \cap T_j;\; for \;\; some \;\; j \in \{1, \ldots, s\} \end{cases}$$

COMB INEQUALITIES



$$x(E(H)) + \sum_{i=1}^{s} x(E(T_i)) - |H| - \sum_{i=1}^{s} |T_i| + \frac{3s+1}{2} - a(k-1) \leq 0$$

$$\downarrow$$

10 + 6 - 8 - 8 + 5 - 2 = 3 > 0

(Violated constraint !!)

THEOREM : Consider G(N_0, Tk) a graph associated to a lagrangean problem solution. Let E’ Ì Tk satisfying the following properties:

$(i) \quad \forall\ (i,j) \in E',\ \ we\ \ have\ \ x(d(i)) = 1\ \ or\ \ x(d(j)) = 1$

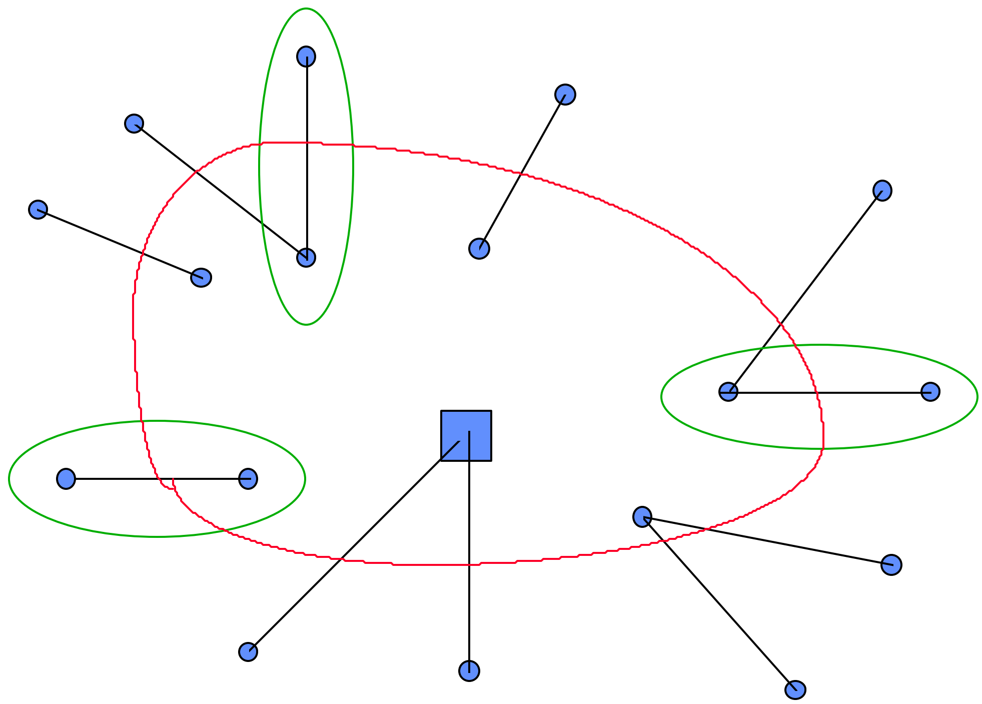
$(ii) \quad \forall\ (i,j) \in E',\ \ we\ \ have\ \ i \neq 0\ \ and\ \ j \neq 0;$

$(iii) \quad If\ two\ \ pairs\ \ of\ \ edges\ (i,j)\ and\ (l,m)\ \ belongs\ \ to\ \ E'$
$then\ \ \{i,j\} \cap \{l,m\} = \varnothing$

$(iv) \quad |E'| \geq 3.$

If E’ Ì Tk satisfies (i)-(iv) then we always find a violated comb.

PROOF:



$$x(E(H))+\sum_{i=1}^{s}x(E(T_i))-|H|-\sum_{i=1}^{s}|T_i|+\frac{3s+1}{2}-a(k-1)\leq 0$$

ni ---> n. of edges with a node of degree 1

$$\underbrace{n + k - ni}_{x(E(H))} + 3 \leq \underbrace{n + 1 - ni}_{|H|} + 6 - 5 + k - 1$$

Then $3 \leq 1$!! (Violated constraint !!)

IDENTIFICATION OF VIOLATED COMBS

$$Let:\ D = \{(0,i) \in Tk \,/\, i \in N\}$$

Heuritic VCOMB;

Begin

- **Construct (if it is possible) a big comb on the graph G(N0,Tk);**

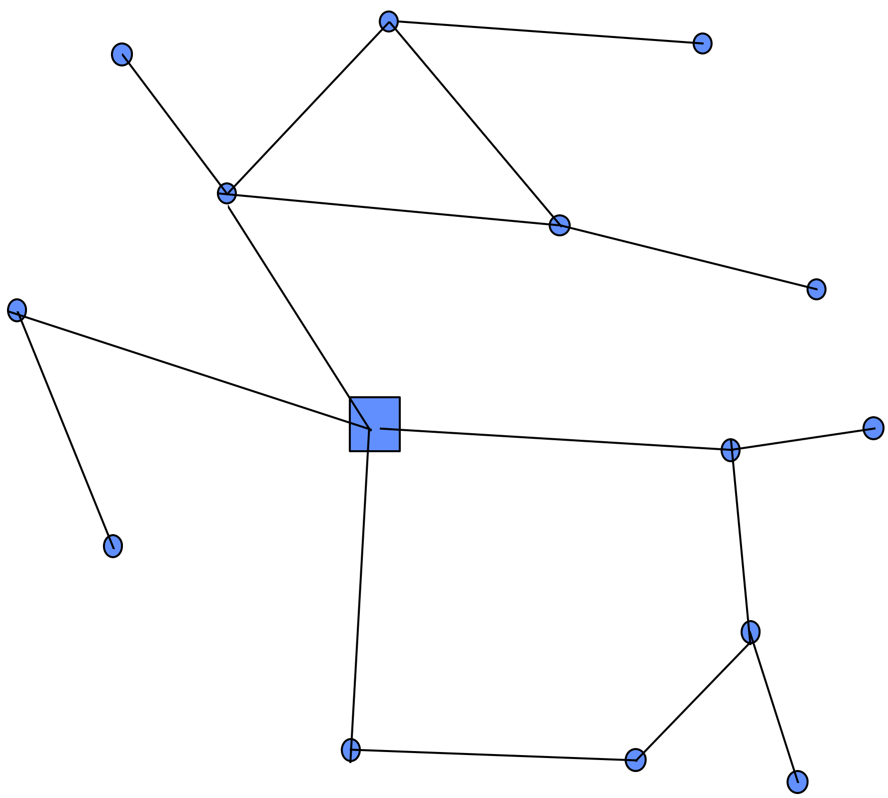
- **Construct G'(N, Tk\D);**
  (we find m connected components)

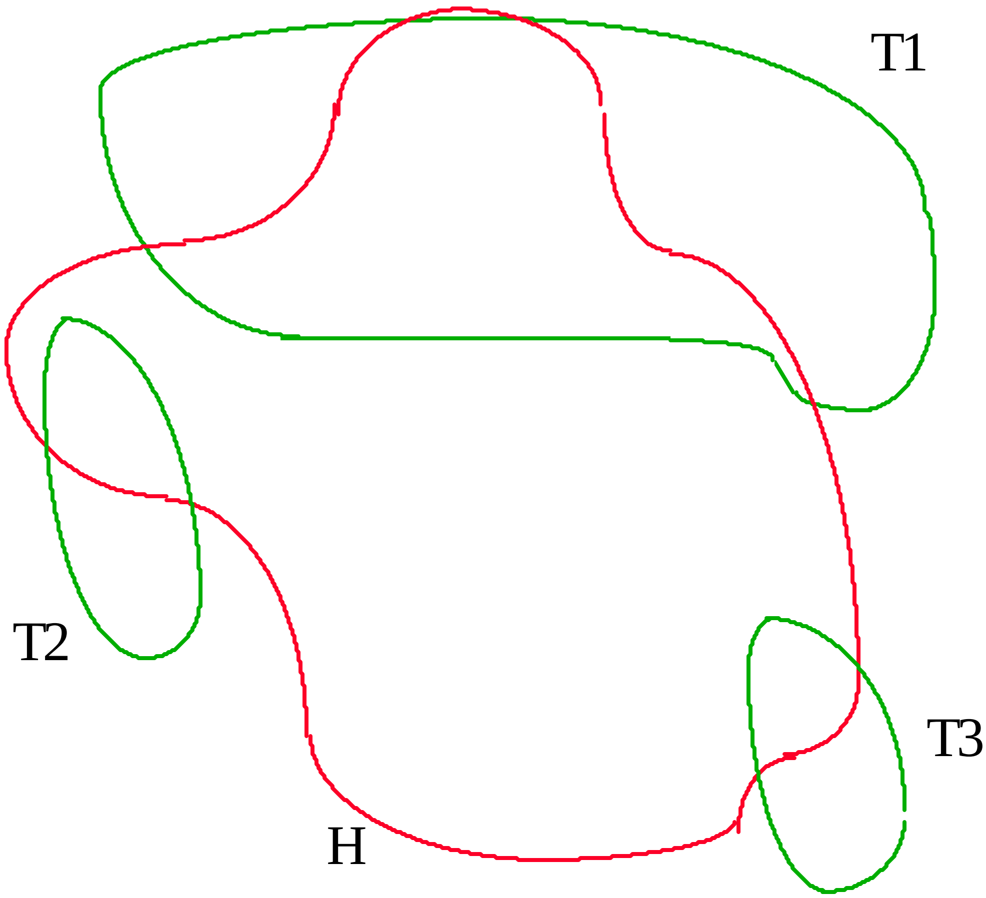
- **Construct (if it is possible) small combs (2-matchings) in each one of the m connected components;**

End.

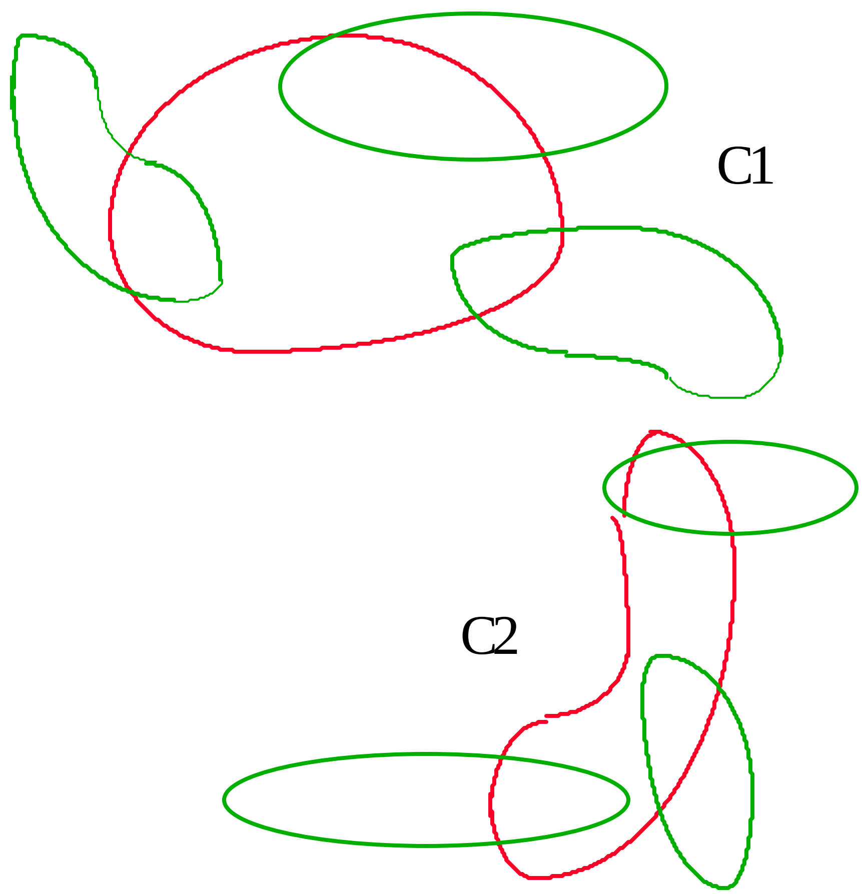
IDENTIFICATION OF VIOLATED COMBS

2-Tree





big comb



small combs

MULTI-STARS INEQUALITIES

(Araque [91])

Let $S_1, S_2, .., S_p$ such that**:**

$$(i) \; S_i \cap S_j = \{v\}$$

$$(ii) \; d(S_i) \leq b; \quad \forall \, i \in \{1,..,s\}$$

$$(iii) \; d(S_i \cup S_j) > b; \quad \forall \, i, j \in \{1,..,p\} \; and \; i \neq j$$

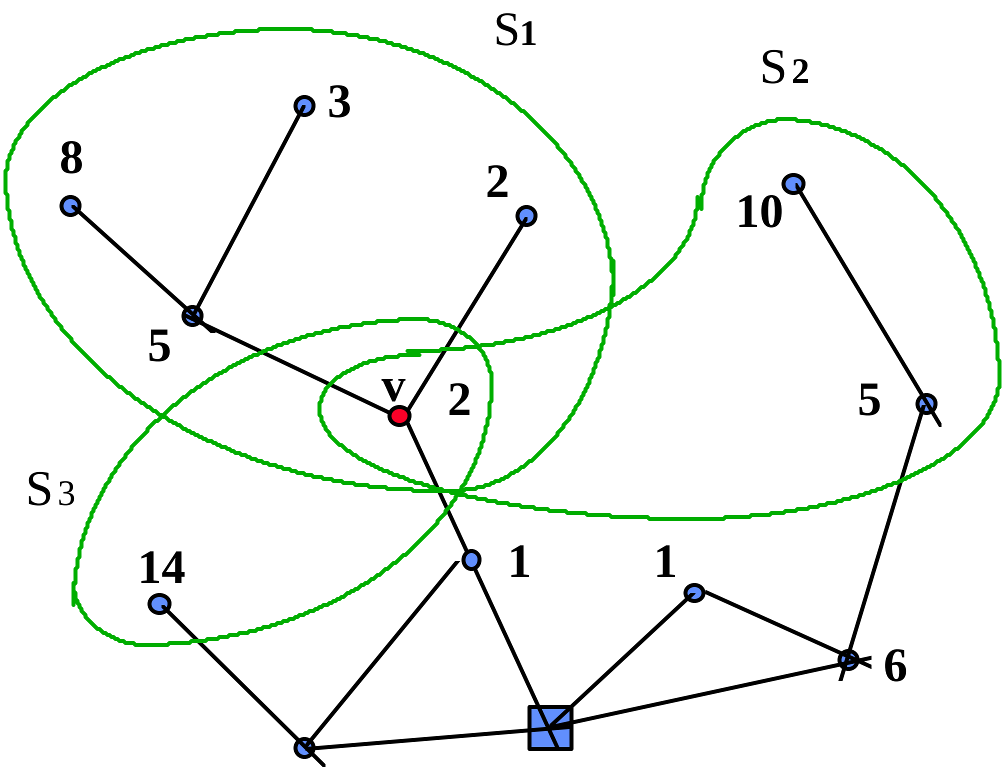
Then

$$\sum_{i=1}^{p} x(d(S_i)) \geq 4p - 2$$

is a valid constraint for VRP polytope

IDENTIFICATION OF MULTI-STARS

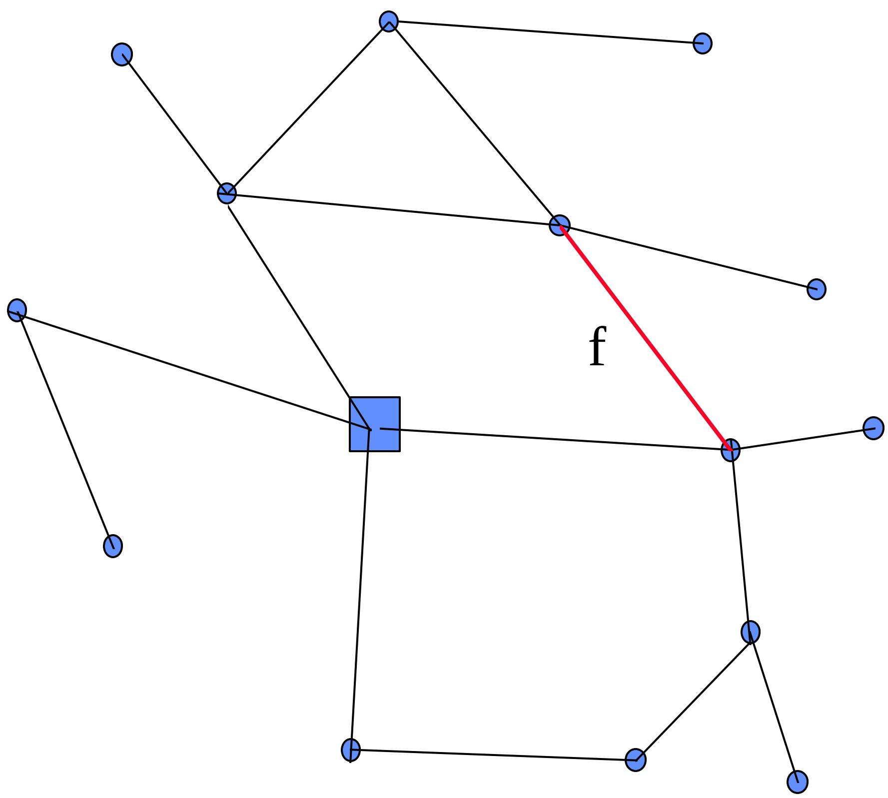
b = 32



d(S1) = 20, d(S2) = 17, d(S3) = 16

FIXING VARIABLES

$f \notin T_K$ and not incidente on the depot

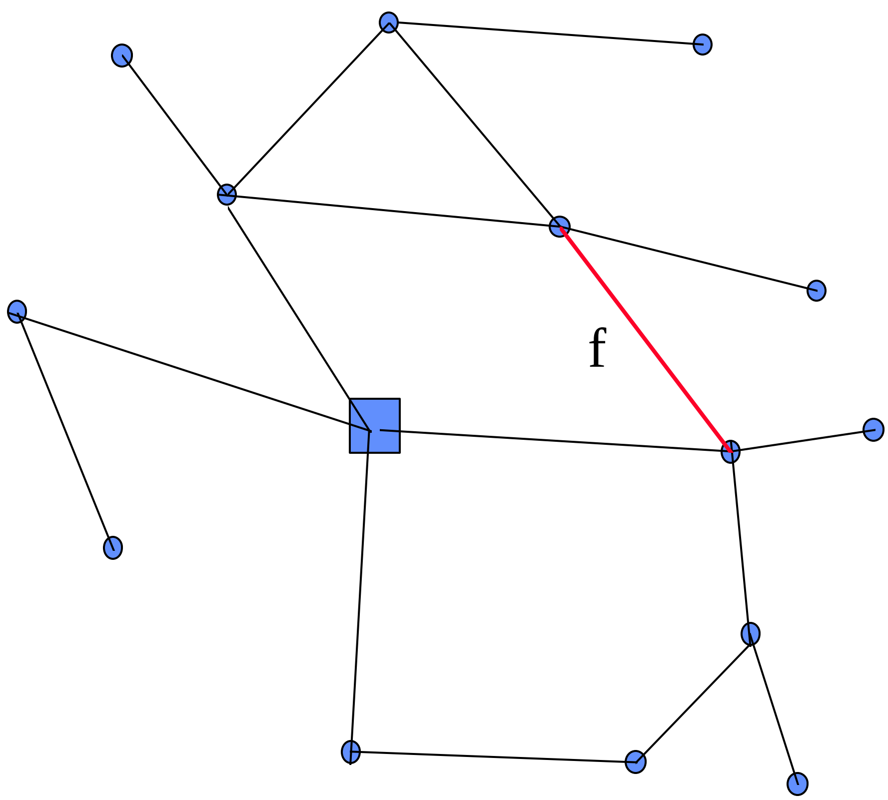


Problem: **(Fixing zeros)**

Find a minimum cost K-tree T^* with degree 2K on the depot and $f \in T^*$.

FIXING VARIABLES

$f \notin T_K$ and not incident on the depot

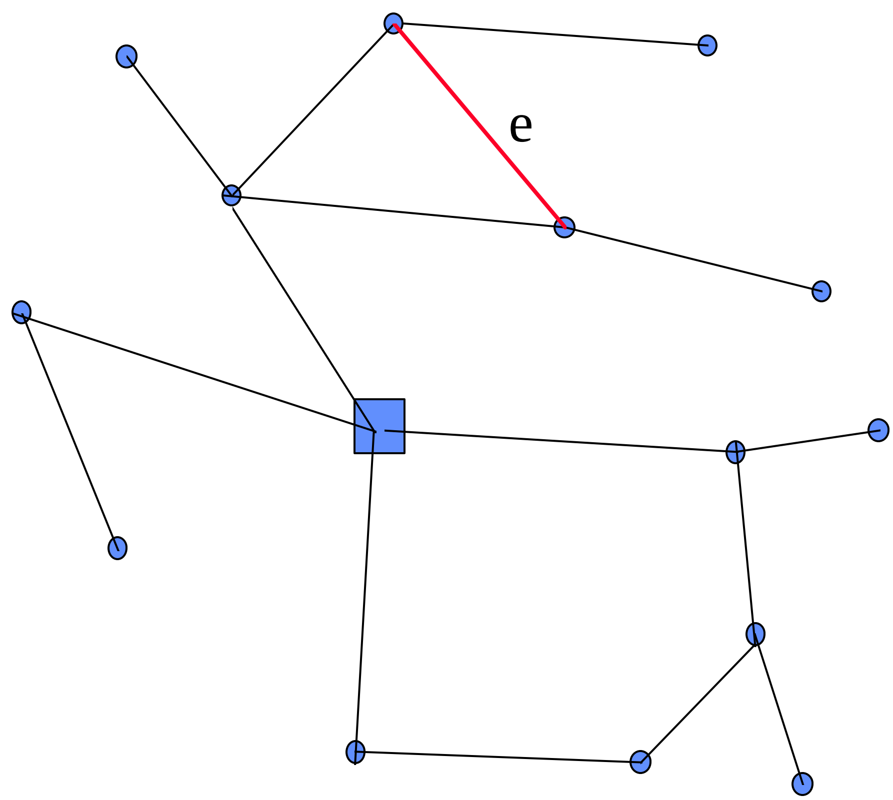


Problem: **(Fixing zero's)**

Find a minimum cost K-tree T* with degree 2K on the depot and $f \in T^*$.

FIXING VARIABLES

e Î T_K and not incident on the depot



Problem: **(Fixing one's)**

Find a minimum cost K-tree T* with degree 2K on the depot and f Ï T*.

FIXING VARIABLES

(Fixing zeros)

Procedure: (Given $\bar{f} \notin T_k$)

Begin

for (each e $\in T_K$) **do**

If ((e, $\bar{f}$) defines an adm. exchange) **then**

- $T_K' = T_K - e \cup \{\bar{f}\};$

- compute minimum adm. exchange (e_0, f_0) in T_K'

end if;

end for;

for (each $e_0 \in T_k$) **do**

If ((e_0 **,** $\bar{f}$) defines an adm. exchange) **then**

- $T_K' = T_K - e_0 \cup \{\bar{f}\};$

- compute min. admissible exchange (e_0, f_0) in T_K'

end if;

end for;

end.

COMPUTATIONAL RESULTS

	Lower Bounds			
Prob.	**Fisher (F)**	**Subt. (S)**	**S+C (SC)**	**S+C+M (SCM)**
c50.dat	507.09	513.51	514.17	514.21
c75.dat	755.50	766.07	762.58	764.09
c100.dat	785.86	792.47	788.75	791.84
c150.dat	932.68	953.66	945.55	952.60
c199.dat	1096.72	1150.23	1128.64	1138.81
c100b.dat	817.77	817.56	817.57	816.68
f44.dat	720.76	723.37	722.03	721.84
f71.dat	237.76	238.65	238.19	238.64
f134.dat	1133.73	1154.45	1147.39	1138.18

Remarks:

- **c problems:** 3000 iterations
- **f problems:** 2000 iterations
- **Reduction :** 75% (50 iterations)
- **euclidean distance**

COMPUTATIONAL RESULTS

Prob.	Lower Bounds Subt (S)	S+C (SC)	S+C+ M (SCM)	Upper Bound
tai75c.dat	1251.85	1247.09	1256.21	1291.01
tai75d.dat	1347.33	1347.65	1344.51	1365.42
tai100a.dat	1939.52	1931.79	1927.91	2047.90
tai100b.dat	1856.3	1850.54	1855.33	1940.61
tai100c.dat	1372.18	1374.39	1358.55	1407.44
tai100d.dat	1475.13	1465.68	1469.89	1581.25
c120.dat	1008.64	1005.72	1002.05	1042.11

Remarks:

- **3000 iterations (subgradient procedure)**
- **Reduction : 75% (50 iterations)**
- **euclidean distance**

FIXING VARIABLES

Problems	LB (S)	Best UB	number of 0´s	number of 1´s
eil22.dat	375.28	375.28	91.19%	100%
eil23.dat	568.56	568.56	94.34%	100%
c50.dat	513.32	524.61	70.25%	0%
c75.dat	768.81	835.26	0%	0%
c100.dat	792.43	826.14	23.75%	0%
c100b.dat	818.07	819.56	93.89%	1.11%
c120.dat	1008.01	1042.11	8.88%	0%
f44.dat	723.24	723.54	93.49%	32.50%
f71.dat	238.63	241.97	76.84%	0%
tai75c.dat	1238.48	1291.01	4.65%	0%
tai75d.dat	1346.56	1365.46	56.11%	0%
tai100a.dat	1934.45	2047.90	0%	0%
tai100b.dat	1851.35	1940.61	0%	0%
tai100c.dat	1372.27	1406.20	9.71%	0%
tai100d.dat	1476.06	1581.25	0%	0%

- reduction: 75% (50 iterations)
- euclidean distance

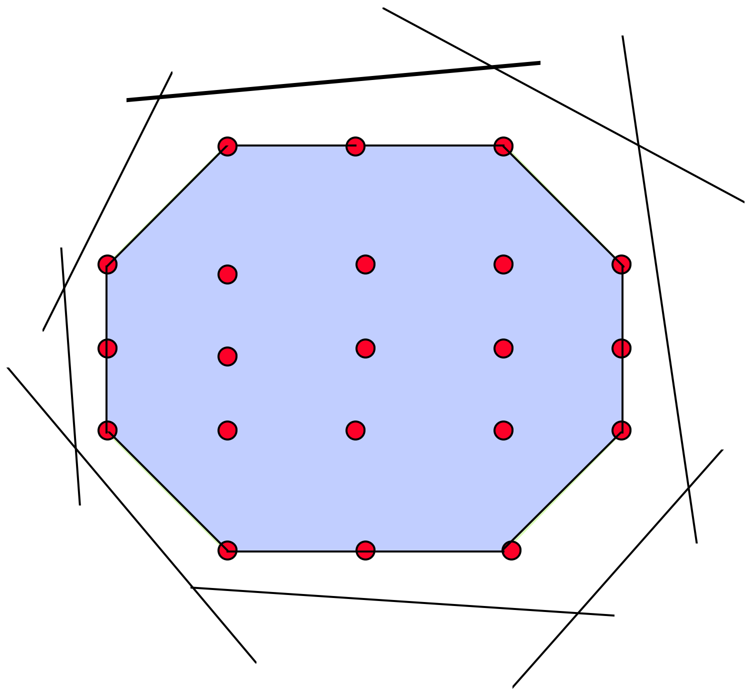
COMPUTATIONAL RESULTS

Prob.	Best UB	Subt. (S)	S+C (SC)	S+C+M (SCM)
eil22.vrp	374	374	374	374
eil23.vrp	569	569	569	569
eil33.vrp	835	830	830	831
c50.dat	521	510	511	512
c75.dat	830	759	761	762
c100.dat	815	784	781	782
c150.dat	1015	938	939	943
c199.dat	1280	1138	1121	1120
c100b.dat	820	819	820	818
f44.dat	724	723	723	722
f71.dat	237	233	233	233
f134.dat	1162	1152	1153	1126

Remarks:

- c, e problems: 3000 iterations
- f problems: 2000 iterations
- Reduction : 75% (50 iterations)
- normalized distance

STRONGER FORMULATIONS
(Vehicle Routing)

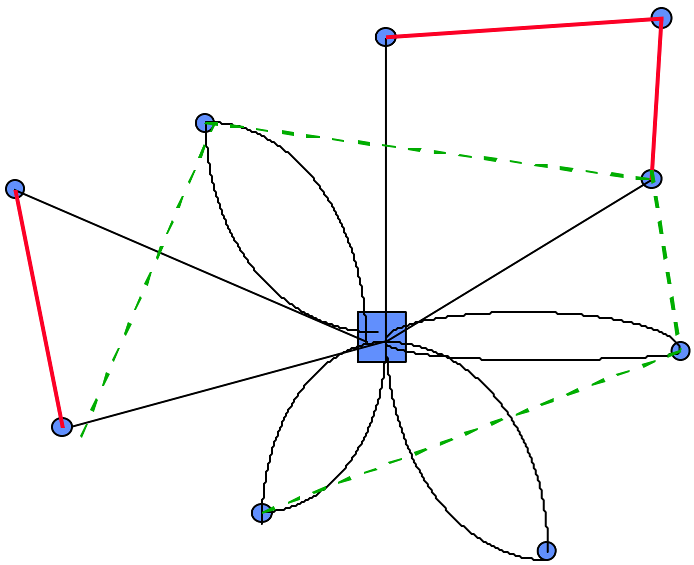


Capacity constraint
Tightened capacity constraint (Fisher[94])
Combs (Cornuéjols & Harche [93])
Multi-stars (Araque [91])
....

CONTRIBUTIONS TO K-TREE RELAXATION APPLIED TO VRP

- **different approach to identify violated constraints (sub-tours, combs and multi-stars)**
- **reduced costs**
- **fixing variables**
- **new lagrangean Clark & Wright heuristic**

Lagrangean Clarke&Wright heuristic



variables fixed in one
variables fixed in zero

Let $E_k(N_0)$ *be the set of non fixed edges*

Positive savings:

$$s_{ij} = \bar{c}_{0i} + \bar{c}_{0j} - \bar{c}_{ij} \qquad \forall (i, j) \in E_k(N_0)$$

LAGRANGEAN CLARCKE&WRIGHT HEURISTIC

Prob.	Best UB	Subt. (S)	S+C (SC)	S+C+M (SCM)
c50.dat	524.61	553.55	548.04	554.60
c75.dat	835.26	860.87	857.02	863.57
c100.dat	826.14	854.22	851.85	855.43
c150.dat	1028.42	1103.36	1097.93	1079.10
c199.dat	1294.25	1374.88	1382.53	1386.83
c100b.dat	819.56	819.56*	823.86	821.11
f44.dat	723.54	723.54*	723.54*	723.54*
f71.dat	241.97	248.25	253.58	248.54
f134.dat	1162.96	1192.24	1197.30	1197.30

Remarks:

- Reduction : 75% (50 iterations)
- euclidean distance

SUBGRADIENT PROCEDURE CONSTRAINTS GENERATION

```
Begin

    do
        - Lower bound  zlb (K-Tree);
        - Upper bound zub (lag. C&W);
        - Update   zub and zlb;
        If   (zlb <  zub)  then
            - Update the active set A;
            - Add violated partitions to A;
            - Update lagrangean multiplers
            - Exclude some partitions of  A;
            - Update lagrangean costs;
        else
            - return to Branch & Bound;
    until (stop condition);
end.
```

COMPUTATIONAL RESULTS

Prob.	Fisher (F)	Subt. (S)	S+C (SC)	S+C+M (SCM)
	Lower Bounds			
c50.dat	507.09	511.95	513.66	515.17
c75.dat	755.50	765.48	765.56	765.06
c100.dat	785.86	793.65	795.36	791.52
c150.dat	932.68	950.86	954.61	950.90
c199.dat	1096.72	1149.70	1148.50	
c100b.dat	817.77	817.54	817.66	817.85
f44.dat	720.76	721.20	722.55	
f71.dat	237.76			
f134.dat	1133.73			

Remarks:

- **c problems:** 3000 iterations
- **f problems:** 2000 iterations
- **Reduction :** 85% (50 iterations)
- **euclidean distance**