III Workshop on Efficient and Experimental Algorithms

An Improved Derandomized Approximation Algorithm for the Max-controlled Set Problem

Carlos Alberto Martinhon
Universidade Federal Fluminense- IC/UFF
mart@dcc.ic.uff.br

Fábio Protti
Universidade Federal do Rio de Janeiro- IM/NCE
fabiop@nce.ufrj.br

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Topics

a) Basic Concepts;

b) The Max Controlled Set Problem;

c) A 1/2-approximation procedure
(Makino, Yamashita, Kameda [2002])

d) The Parameterized CSP;

e) An Improved Randomized Rounding Procedure;

f) A derandomized algorithm

g) Conclusions

Basic Concepts

Consider G=(V,E) a non-oriented graph and M Í V.

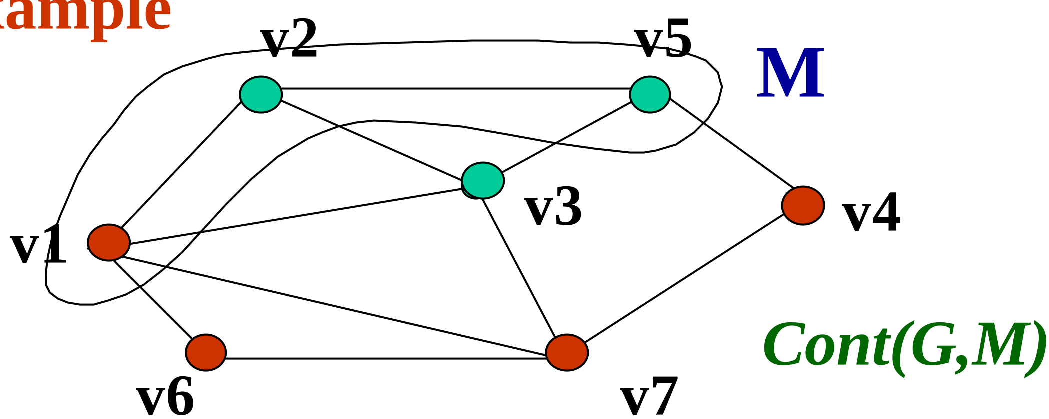
Definition 1: (Neighborhood of v Î V)

N_G[v] = {w Î V | (w,v) Î E} U {v}

Definition 2: v is controlled by M Í V

Û |N_G(v)ÇM| ³ |N_G(v)|/2

Example



M defines a *monopoly* if *Cont(G,M)*=V

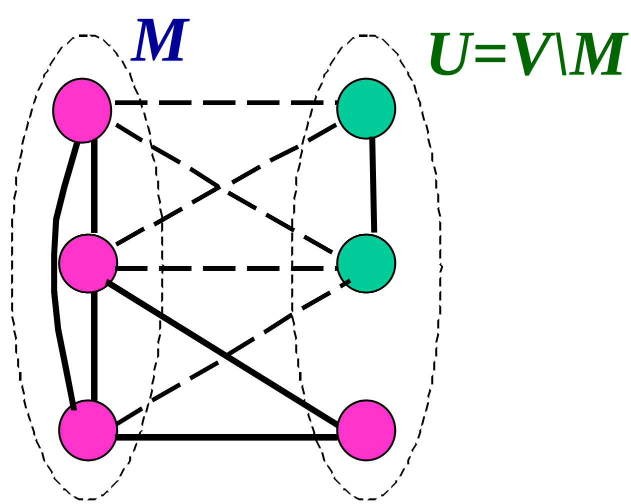
The Monopoly Verification Problem MVP

Input: $G_1=(V,E_1)$ and $G_2=(V,E_2)$ s.t. E_1 Í E_2 and M Í V

Question: $ $G=(V,E)$ s.t. E_1 Í E Í E_2 and M is monopoly in G ?

Output: Yes or No

$G_2=(V,E_2)$

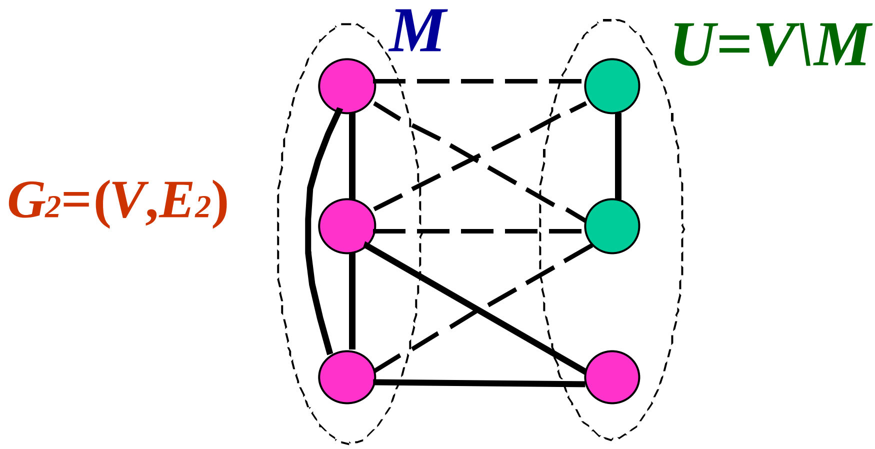


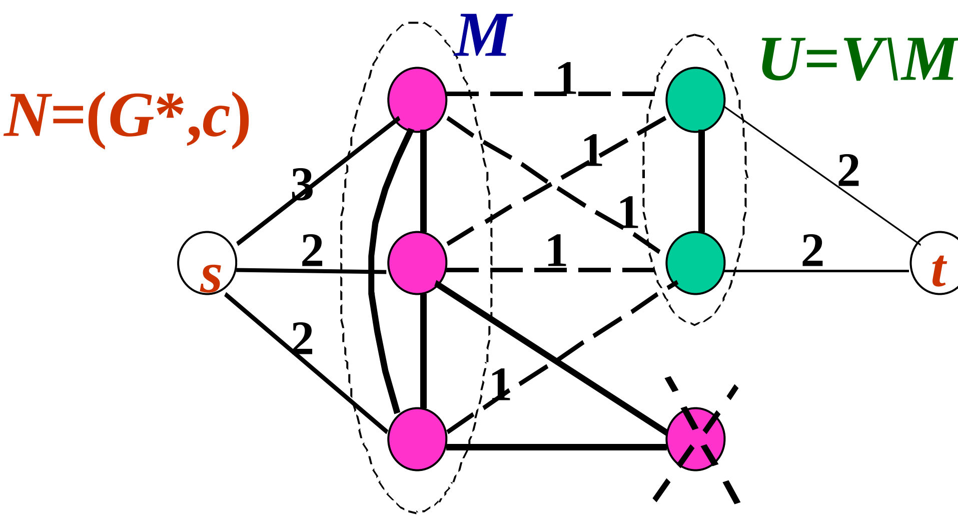
We first apply some reduction rules

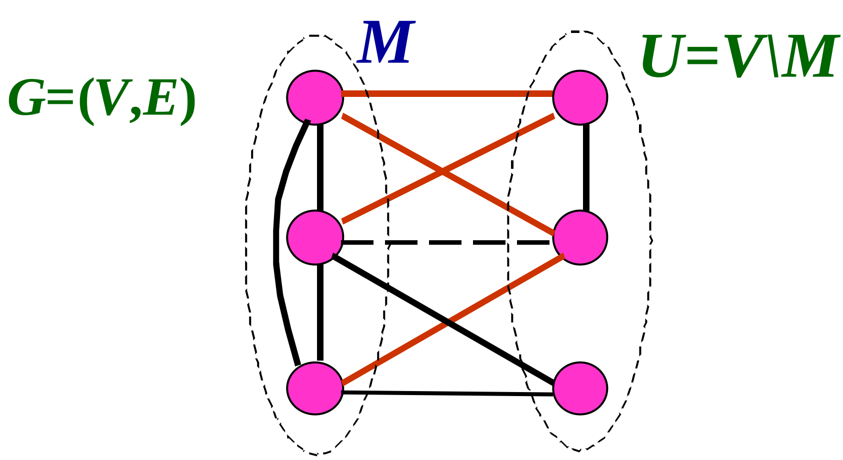
Polynomial time ® $\tilde{O}\left(\min\{(n+m_2)^{3/2}, n^{2/3}(n+m_2)\}\right)$

Makino, Yamashita, Kameda [2002]

The Monopoly Verification Problem

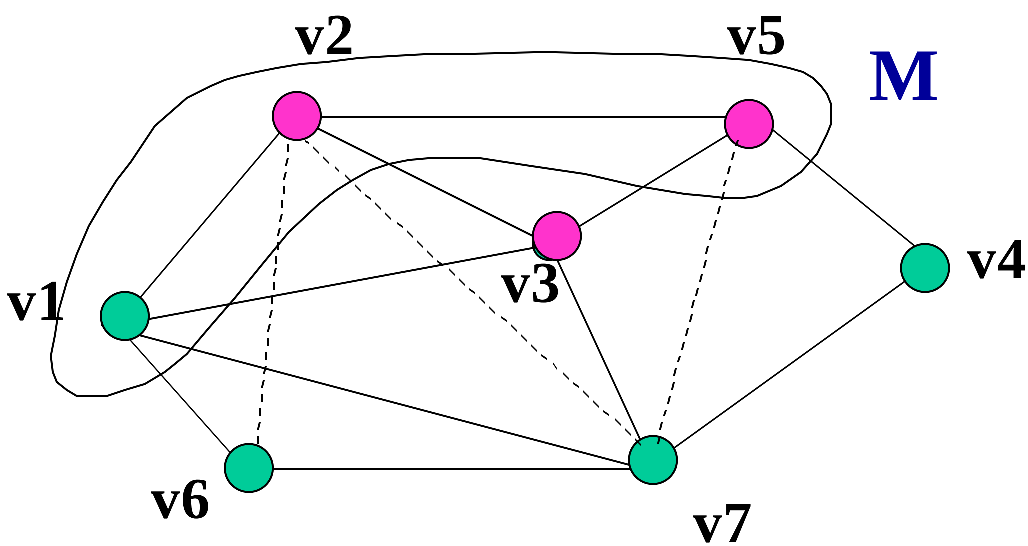




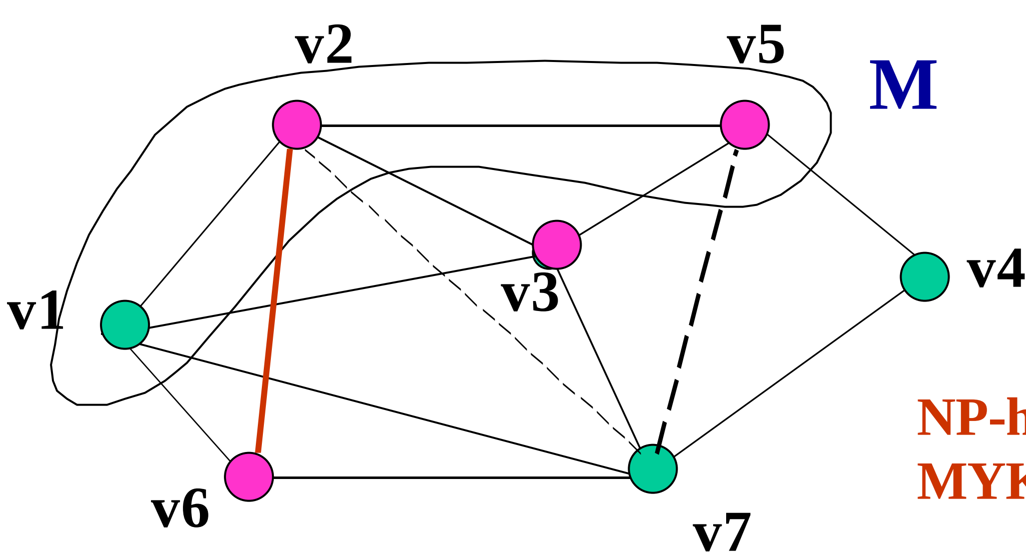


The Max-Controlled Set Problem

If the answer to the MVP is No!



Cont(G,M) ®



NP-hard !!
MYK[2002]

A 1/2-Approximation Algorithm for the MCSP

(Makino, Yamashita, Kameda [2002])

Begin

1. w_1 ¬ **Compute |*Cont(G,M)*| by removing from *G*2 the edge set *E*2*E*1**
2. w_2 ¬ **Compute |*Cont(G,M)*| by adding to *G*1 the edge set *E*2*E*1**
3. **Return** $z \leftarrow \max\{w_1, w_2\}$

end.

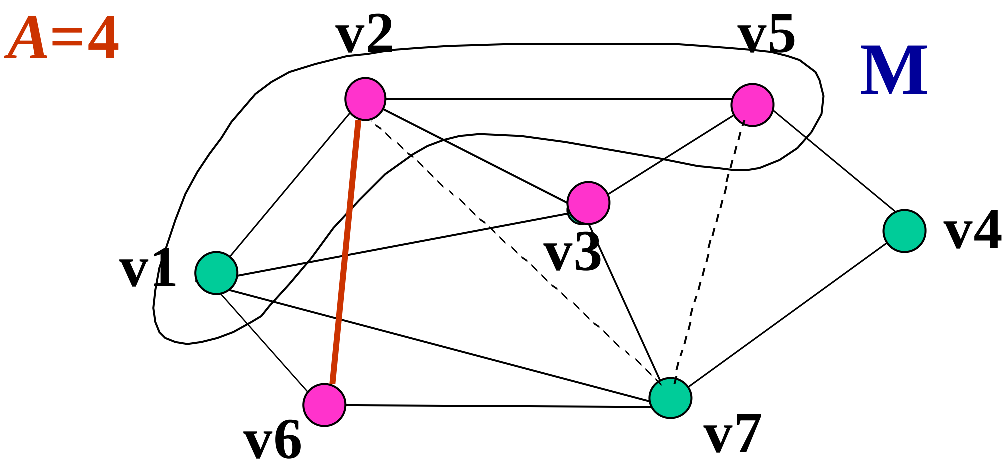
Theorem: The deterministic MYK procedure has performance ratio equal to 1/2 for the MCSP

Parameterized CSP

Input: $G_1=(V,E_1)$ and $G_2=(V,E_2)$ such that $E_1 \subseteq E_2$, $M \subseteq V$.

Parameter: $A>0$

Question: $\exists\, G=(V,E)$ s.t. $E_1 \subseteq E \subseteq E_2$ s.t M controls at least A vertices?

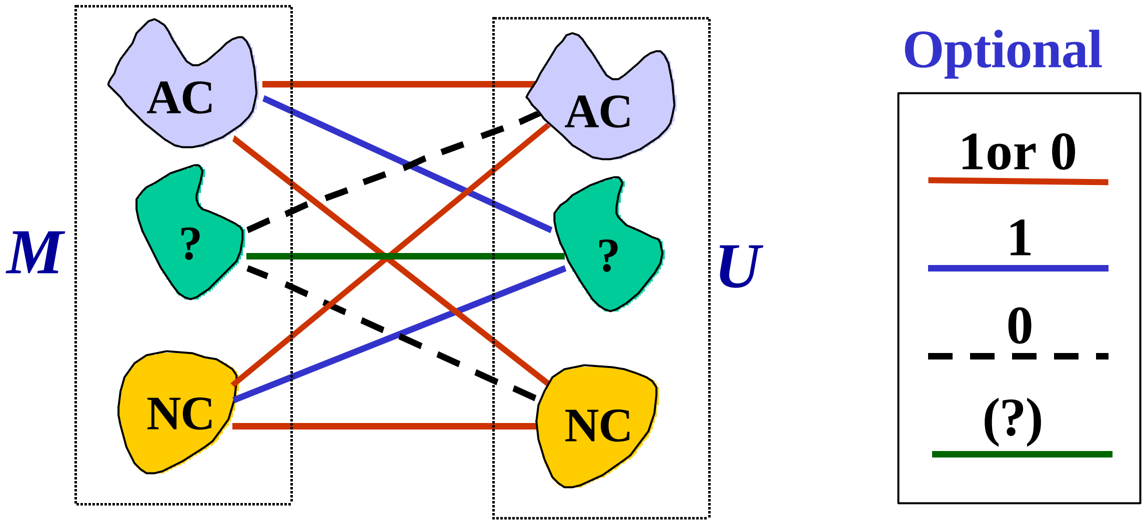


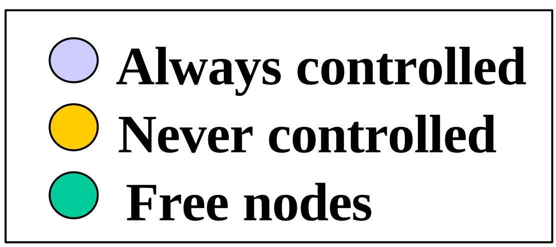
Polynomial time $\rightarrow$ $O(n^A)$

Open problem: The Parameterized CSP is FPT?

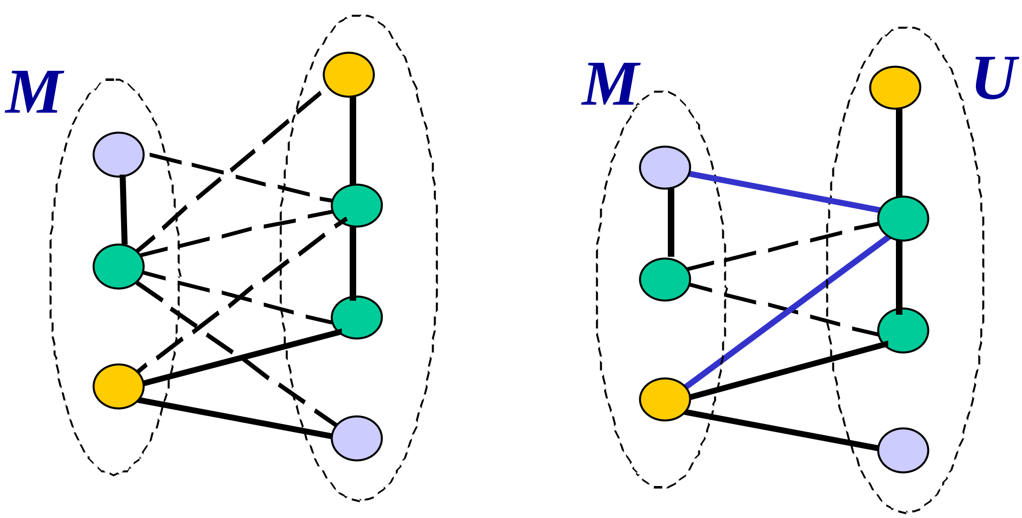
Note: Is there any algorithm for the Parameterized CSP with runtime $O(f(A).n^a)$ *for some* $a>0$?

New Reduction Rules





Example:



Randomized Rounding
(Raghavan&Thompson[1987])

Generic Procedure

x* ← **Linear relaxation;**

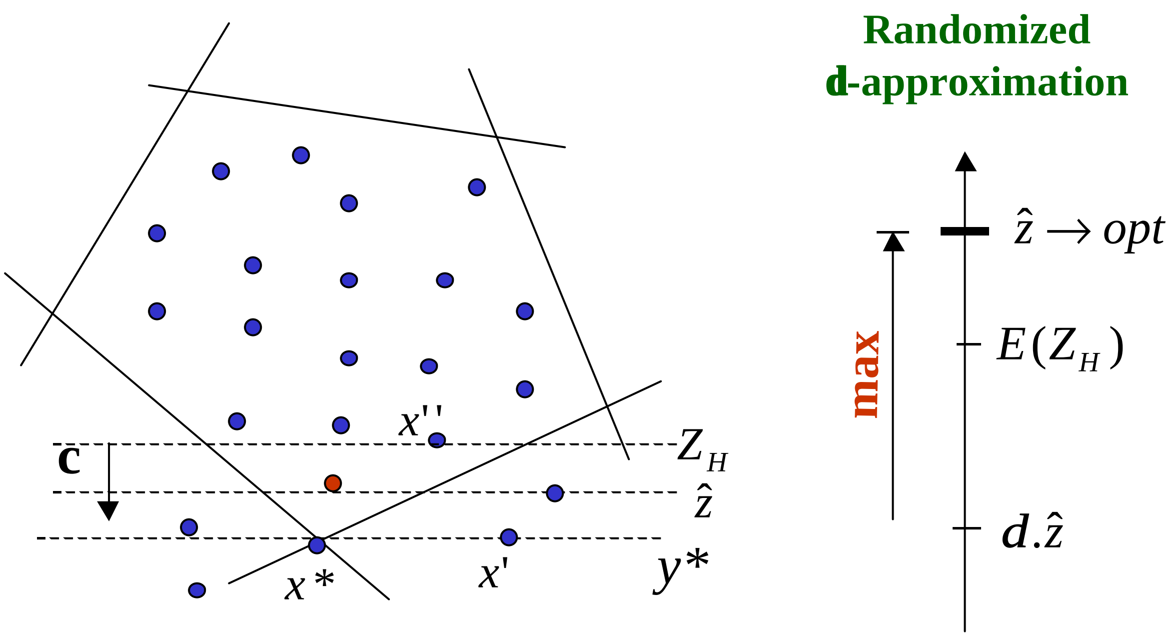
x’ ← **Linear (or nonlinear) rounding;**

$$\Pr(x_j = 1) = f(x_j^*);$$

$$\Pr(x_j = 0) = 1 - f(x_j^*) \quad \text{for } j = 1,..,n$$

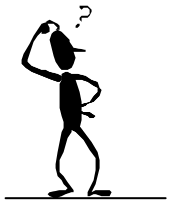
x’’ ← **“Derandomize” x’;**

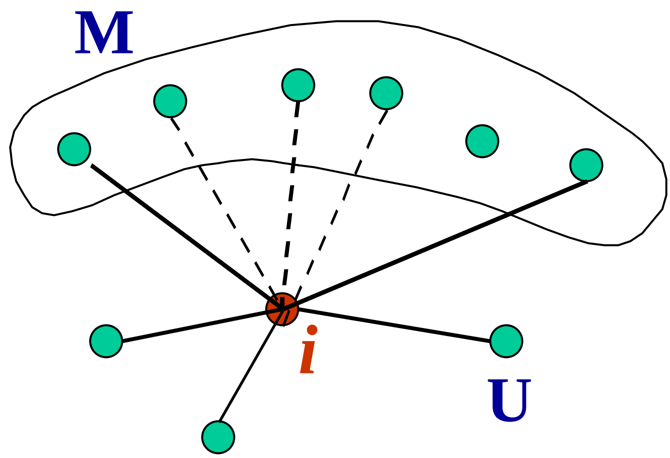
end.



Feasible set **Objective function**

A Linear Integer Programming formulation for the MCSP





$$a_{ij} = \begin{cases} 1, & if \ \ (i,j) \in E_2 \\ 0, & otherwise \end{cases}$$

$$|V| = n$$

$$\hat{z} = \max\left(\sum_{i \in V} z_i \right)$$

$$s.t. \begin{cases} \sum_{j \in M} \left(\dfrac{a_{ij}}{n} \right) x_{ij} - \sum_{j \in V} \left(\dfrac{a_{ij}}{2n} \right) x_{ij} + 1 \geq z_i \ , & \forall \ i \in V \\ x_{ij} = 1, & \forall \ (i,j) \in E_1 \\ x_{ij} \in \{0,1\}, & \forall \ (i,j) \in E_2 \setminus E_1 \\ z_i \in \{0,1\}, & \forall \ i \in V \end{cases}$$

y* ® value of the linear relaxation

Randomized Rounding Procedure

Algorithm 2: (Randomized MCS)

1. z_1 ¬ Apply the MYK deteministic procedure;
2. Solve the linear relaxation and return x* and y*;
3. Using an “oracle” choose k Î(1,2];
4. Compute:

$$A(k) = \frac{2k(k-1)+1+\sqrt{4k(k-1)+1}}{2(k-1)^2}$$

5. If y^* £ $A(k)$ do

   z_H ¬ Apply the Parameterized CSP and Stop

   else

$$\Pr(\bar{x}_{ij} = 1) = x^*_{ij}$$

$$\Pr(\bar{x}_i = 0) = 1 - x^*_{ij}$$

6. Compute z_2 using the integer solution $\bar{x}$
7. Return $z_H \leftarrow \max\{z_1, z_2\}$

end.

Approximation Analysis - MCSP

Definitions: X_{ij} **Î {0,1}, for every (*i,j*)**
Z_i = 1, if *i* Î *V* is controlled by *M*
Z_i = 0, otherwise

$$Z_H = \sum_{i \in V} Z_i \quad \longrightarrow$$ **value of the randomized solution**

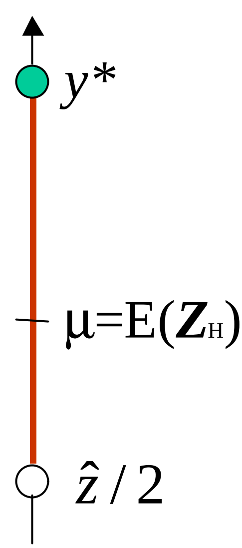
From our Randomized MCS algorithm and from the L.P. relaxation:

$$\hat{z}/2 \leq E(Z_H) \leq y^*$$

Consider (w.l.o.g):

$$m = E(Z_H) = y^*/b$$
$$for\ some\ \ b \in [1,2)$$

Problem: $b = ?$



Approximation Analysis (cont.)

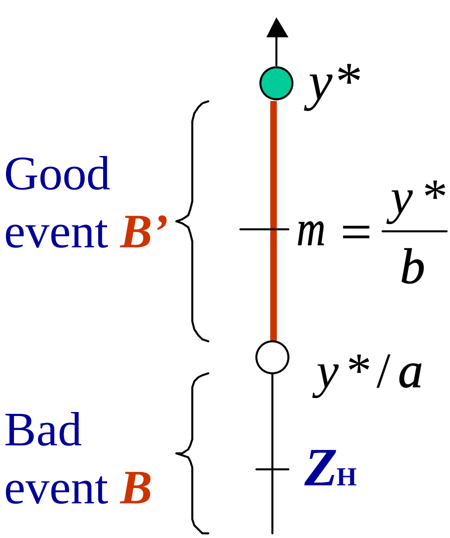
Consider a bad event

$$B \equiv \left(Z_H < \frac{y^*}{a} \right) \quad for \;\; some \;\; a > 1$$

Then, Z_H defines a randomized 1/a-approximation solution for the MCSP if B’ ¹ Æ (complementary event).

> **Question:** **How small a value for a can we achieve while guaranteeing good events B’ ¹ Æ ?**

For our purposes, consider a randomized 1/µ -approx. algorithm for the MCSP for some a Î (b , k) with k£2.



$$\exists \; a \in (b, k) \;\; s.t. \;\; B' \neq \emptyset ?$$

$$\Pr(B) < 1 \quad \Leftrightarrow \quad \Pr(B') > 0$$

$$\Pr(B) = \Pr\left(Z_H < \frac{y^*}{a} \right) = ?$$

Approximation Analysis (cont.)

Since: y*=βμ

$$\Pr(B) = \Pr\left(Z_H < \frac{y^*}{a} \right) = \Pr\left(Z_H < \frac{bm}{a} \right) = \Pr\left(Z_H < (1-d)m\right)$$

where: $d = 1 - b/a > 0$

Theorem: Lower Chernoff-Hoeffding+Negative Association

Let $Z_1, Z_2, \ldots, Z_n$ be negatively assotiated Poisson trials s.t, for 1 £ *i* £ *n*, $\Pr(Z_i = 1) = p_i$, where $0 < p_i < 1$. Then, for

$$Z_H = \sum_{i=1}^{n} Z_i, \quad m = E(Z_H) = \sum_{i=1}^{n} p_i \quad and \quad 0 < d \leq 1,$$

we have that:

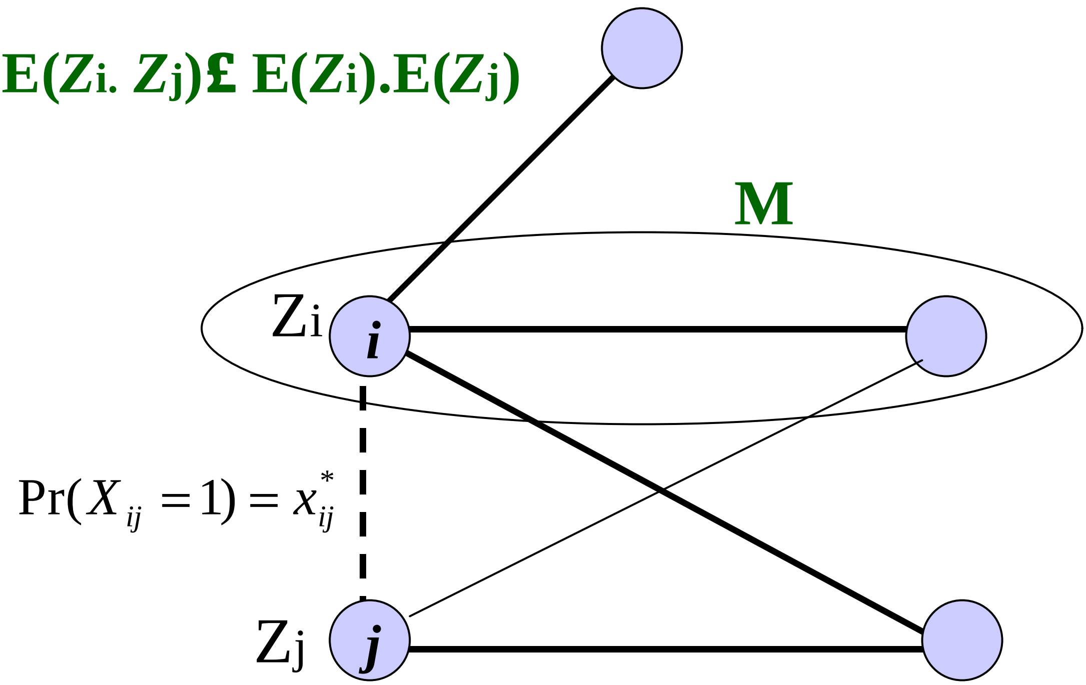
$$\Pr\left(Z_H < (1-d)m\right) < \frac{1}{\exp^{(md^2/2)}}$$

Theorem: The random variables Z_i, for all *i* Î *V* are negatively associted

Negative Assotiation

Z_i and Z_j are negatively associated !!

E(Zi. Zj)£ E(Zi).E(Zj)



$$\Pr(Z_i = 1) = \Pr(X_{ij} = 0) = 1 - x_{ij}^*$$

but

$$\Pr(Z_i = 1 \mid Z_j = 1) = 0 \;!!$$

Approximation Analysis (cont.)

Thus, by the lower CH bound and assuming $m > \frac{\hat{z}}{2}$

$$\Pr(B) < \frac{1}{\exp\left((1-b/a).(m/2)\right)} < \frac{1}{\exp\left((1-b/a).(\hat{z}/4)\right)}$$

$$< \frac{1}{\exp\left((1-b/a).(\hat{z}/4)\right) - \exp(1/4)} < 1 \qquad \text{(I)}$$

Question: **$ a Î (b, k) for some b ³ 1 and k £ 2 ?**

From (I):

$$(\hat{z}-2)a^2 - (2\hat{z}b)a + b^2\hat{z} > 0, \quad \textit{for some } a \in (b, k)$$

The values

$$a' = \frac{b(\hat{z}-\sqrt{2\hat{z}})}{\hat{z}-2} \quad \textit{and} \quad a'' = \frac{b(\hat{z}+\sqrt{2\hat{z}})}{\hat{z}-2}$$

are roots of equation: $(\hat{z}-2)a^2 - (2\hat{z}b)a + b^2\hat{z} = 0.$

Approximation Analysis (cont.)

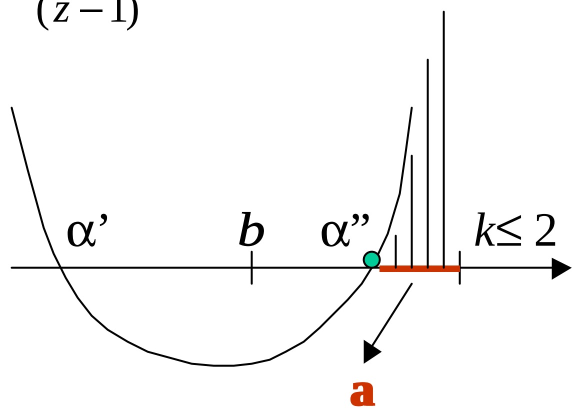
Since, we are considering a > b, it follows that:

$$\frac{\hat{z} \pm \sqrt{\hat{z}}}{(\hat{z}-1)} > 1 \quad \Rightarrow \quad \frac{\hat{z} + \sqrt{\hat{z}}}{(\hat{z}-1)} > 1 \quad for\ \ \hat{z} > 1.$$

Then:

$$a'' = \frac{b(\hat{z} + \sqrt{\hat{z}})}{\hat{z}-1}$$

$$with\ \ \hat{z} - 2 > 0$$



In addition: a” < *k*, b=*y/m and m <y*. Then:**

$$a'' = \frac{y^*(\hat{z} + \sqrt{\hat{z}})}{m(\hat{z}-1)} < k \quad \Rightarrow \quad \frac{y^*(\hat{z} + \sqrt{\hat{z}})}{k(\hat{z}-1)} < E(Z_H) < y^*. \qquad \text{(II)}$$

Therefore, from (II):

A) $$\frac{y^*(\hat{z} + \sqrt{\hat{z}})}{k(\hat{z}-1)} < y^* \quad \Rightarrow \quad \hat{z} + \sqrt{\hat{z}} < k(\hat{z}-1) \ \ for\ every\ \ \hat{z} > A(k).$$

where:

$$A(k) = \frac{2k(k-1) + 1 + \sqrt{4k(k-1)+1}}{2(k-1)^2}$$

Approximation Analysis (cont.)

B)

$$\left(\frac{1}{k}+\frac{2+\sqrt{2\hat{z}}}{k(\hat{z}-2)}\right)y^* = \left(\frac{\hat{z}+\sqrt{2\hat{z}}}{k(\hat{z}-2)}\right)y^* < E(Z_H)$$

Now:

$$\frac{2+\sqrt{2y^*}}{k(y^*-2)} < \frac{2+\sqrt{2\hat{z}}}{k(\hat{z}-2)} \qquad for \quad \hat{z} < y^*$$

Finally:

$$\left(\frac{1}{k}+\frac{2+\sqrt{2y^*}}{k(y^*-2)}\right)\hat{z} < \left(\frac{1}{k}+\frac{2+\sqrt{2\hat{z}}}{k(\hat{z}-2)}\right)y^* < E(Z_H)$$

Theorem: For a given parameter *k* Î (b,2] with *b*=*y*/m*, the Randomized MCS procedure has perfornance ratio equal to:

$$\frac{1}{k}+\frac{2+\sqrt{2y^*}}{k(y^*-2)} \qquad for \ every \ \ y^* > A(k)$$

where

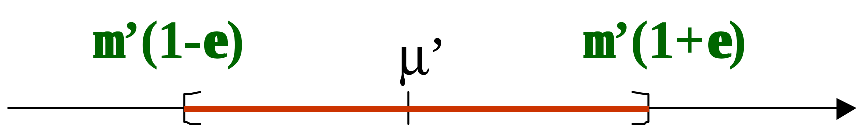
$$A(k) = \frac{2k(k-1)+1+\sqrt{4k(k-1)+1}}{(k-1)^2}$$

Problem: b =y*/m =? and k=?

Approximation Analysis (cont.)

Dagum, Karp, Luby and Ross [2000]

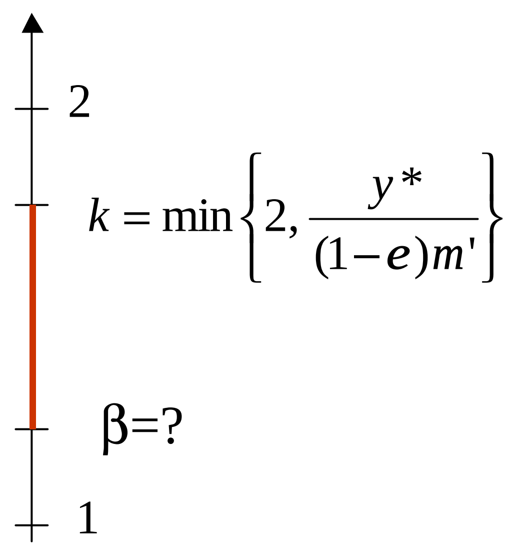
Independent experiments with respect to Z_H, ensures, for a given d and e, an estimator m’ of m=$E(Z_H)$ within a factor 1+e and probability at least 1-d.



m=$E(Z_H)$=?

Contruction of interval (b,k)

$$b = \frac{y^*}{m} < \frac{y^*}{(1-e)m'} = k$$



$$k = \min\left\{2, \frac{y^*}{(1-e)m'}\right\}$$

Approximation Analysis (cont.)

Summary:

Ratio ® $\frac{1}{k}+\frac{1+\sqrt{y^*}}{k(y^*-1)}$ $\quad for\ every\ y^* > A(k)$

with: $A(k)=\frac{2k(k-1)+1+\sqrt{4k(k-1)+1}}{2(k-1)^2}$

1£b<k£2

k	1/k	A(k)
2	0,5	4
1,5	0,66	9
1,3	0,77	18,78
1,1	0,91	220
1,01	0,99	10201

b=y*/m

m =?

Theorem: **For *y**=**q**(*n*) and *k*=2 we have a randomized algorithm with performance ratio equal to**

$$\frac{1}{2}+\frac{1+\sqrt{n}}{2n-2} \quad for\ n>4$$

Derandomization

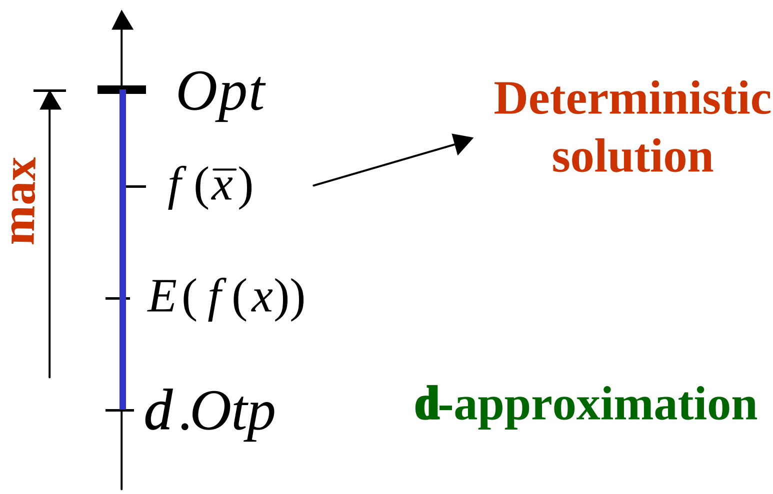
Method of Conditional Expectations

(Spencer)

$$f(x) = \sum_{j=1}^{n} c_j x_j \quad \rightarrow$$

Objetive function

(maximization)



$$E(f(x)) = E(f(x) \mid x_1 = 1).\Pr(x_1 = 1) + E(f(x) \mid x_1 = 0).\Pr(x_1 = 0) \leq$$

$$\leq \max\{E(f(x) \mid x_1 = 1), E(f(x) \mid x_1 = 0)\} = E(f(x) \mid X_1) \leq$$

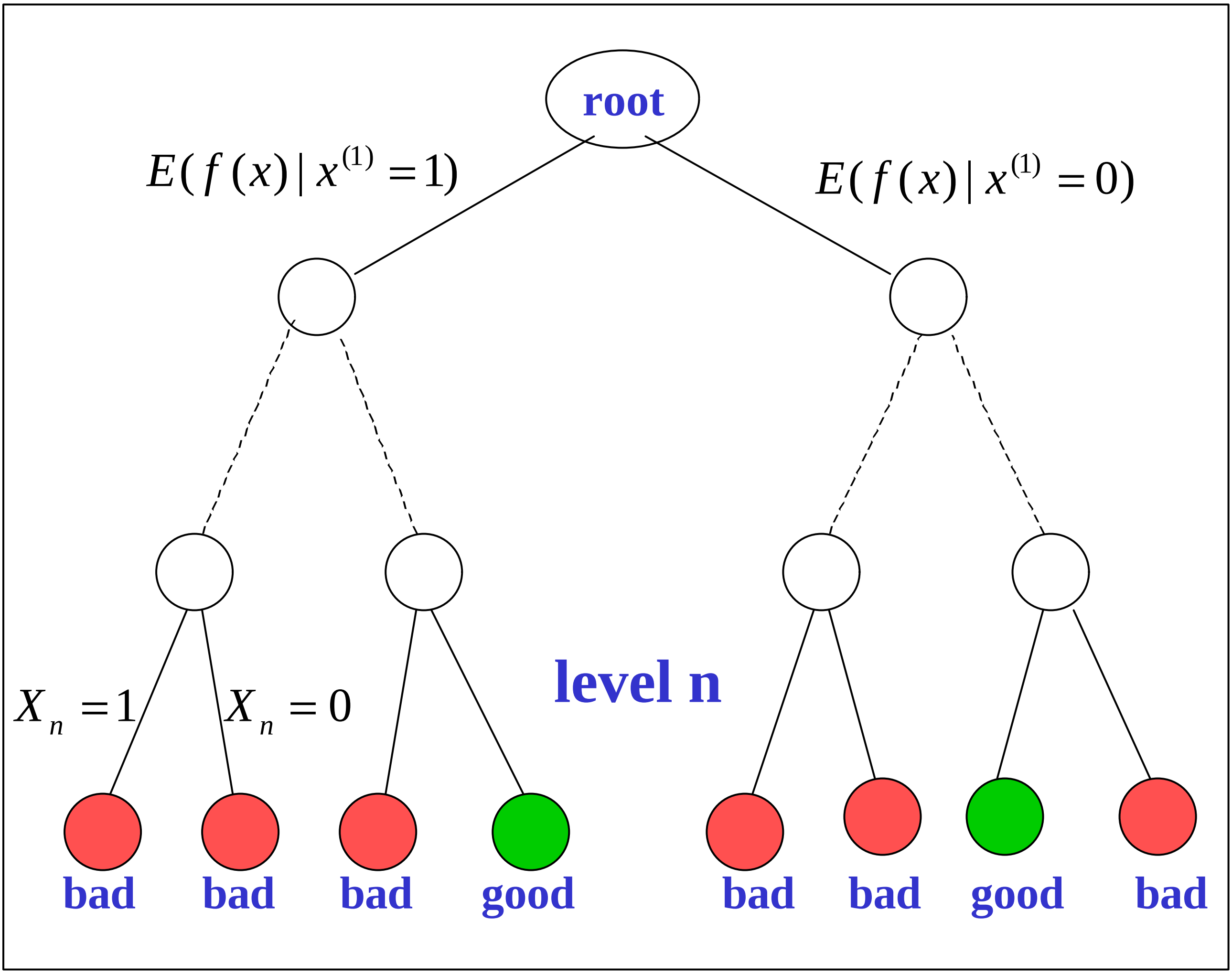
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\leq \max\{E(f(x) \mid X_1, .., X_{n-1}, x_n = 1), E(f(x) \mid X_1, .., X_{n-1}, x_n = 0\}$$

$$= E(f(x) \mid X_1, \ldots, X_n) = \sum_{j=1}^{n} c_j X_j = f(\overline{x})$$

Method of Conditional Expectations

Spencer [1987]



How to walk down the tree to a good leaf in deterministic polinomial time?

A Derandomized Algorithm

From our L.I.P formulation:

$$Z_H = \sum_{i \in V} Z_i \le \sum_{i \in V} \underbrace{\min\left(1; \sum_{j \in M} (a_{ij}/n) X_{ij} - \sum_{j \in V} (a_{ij}/2n) X_{ij} + 1 \right)}_{Y_i}$$

Lemma: Suppose X Î {0,1} and Y Î {0,1} arbitrary random variables: Then:

$$E(\min(X, Y)) \le \min(E(X), E(Y))$$

Then:

$$E(Z_H) = \sum_{i \in V} E(Z_i) \le \sum_{i \in V} \min\left(1; \sum_{j \in M} (a_{ij}/n) x_{ij}^* - \sum_{j \in V} (a_{ij}/2n) x_{ij}^* + 1 \right)$$

$$= \sum_{i \in V} E(Y_i) = \sum_{i \in V} \min\{1, z_i^*\} = y^*$$

$$\text{where} \quad E(X_{ij}) = x_{ij}^* \qquad (III)$$

Notation:

$$x^{(k)} \leftrightarrow X_{ij}, \quad \text{for every } (i, j)$$

$$X^{(k)} \leftrightarrow X_{ij} = 0 \text{ or } 1$$

A Derandomized Algorithm (cont.)

From the method of conditional expectations:

$$E(Z_H) \le \max\left\{ \sum_{i \in V} E(Z_i \mid X^{(1)},\ldots,X^{(k-1)}, x^{(k)} = 0); \sum_{i \in V} E(Z_i \mid X^{(1)},\ldots,X^{(k-1)}, x^{(k)} = 1) \right\}$$

However, for every $i \in V$ and $k \in \{1,\ldots,|E_2 \backslash E_1|\}$

$$\mathrm{E}(Z_i \mid X^{(1)},\ldots, X^{(k-1)}, x^{(k)} = 0 \text{ or } 1) = \Pr(Z_i = 1 \mid X^{(1)},\ldots, X^{(k-1)}, x^{(k)} = 0 \text{ or } 1)$$

Problem: The values

$$\Pr(Z_i = 1 \mid X^{(1)},\ldots, X^{(k-1)}, x^{(k)} = 0 \text{ or } 1)$$

are hard to compute !!

Lemma: Consider *Zi* and *Yi* as above. Then

$$E(Z_i \mid X^{(1)},\ldots,X^{(k-1)}, x^{(k)} = 1 \text{ or } 0) \ge E(Z_i \mid X^{(1)},\ldots,X^{(k-1)}, x^{(k)} = 0 \text{ or } 1)$$

$$\Leftrightarrow$$

$$E(Y_i \mid X^{(1)},\ldots,X^{(k-1)}, x^{(k)} = 1 \text{ or } 0) \ge E(Y_i \mid X^{(1)},\ldots,X^{(k-1)}, x^{(k)} = 0 \text{ or } 1)$$

A Derandomized Procedure

Begin

1. z_1 ¬ **Apply the MYK deteministic procedure**
2. **Solve the linear programming relaxation and return y*;**
3. **If y*£ 4 then**

   Z_H ¬ **Apply the Parameterized MCS and Stop;**

   else

   For k**=1 to** $|E_2 \backslash E_1|$ **do**

   If $E(Y_i \mid X^{(1)},\ldots,X^{(k-1)}, x^{(k)}=1)$ ³ $E(Y_i \mid X^{(1)},\ldots,X^{(k-1)}, x^{(k)}=0)$ **then**

   $X^{(k)} = 1$

   else

   $X^{(k)} = 0$

   Compute z_2 **using the integer solution** X

   Compute: $Z_H \leftarrow \max\{z_1, z_2\}$

4. **Return** Z_H;

end

Complexity $\longrightarrow O(n^4, |E_2 \backslash E_1| n^3 L)$

Conclusions

- **If a good estimation of expectation is obtained in advance (randomized procedure)**

$$\frac{1}{k}+\frac{1+\sqrt{y^*}}{k(y^*-1)} \qquad for\ k \le 2$$

- **If $y^*=\mathbf{q}(n)$ and $k=2$ (deterministic procedure)**

$$\frac{1}{2}+\frac{1+\sqrt{n}}{2n-2}$$

Future Directions

- **Inapproximability results ?**
- **Randomized rounding using Semidefinite Relaxation ?**
- **The Parameterized CSP is FPT?**