Colored trees in edge-colored graphs

A. Abouelaoualim, V.Borozan, Y. Manoussakis, C. Martinhon*, R. Muthu, R. Saad

University of Paris-XI, Orsay LRI, Bât. 490, 91405 Orsay Cedex, France
∗ Inst. of Comp. / Fluminense Federal University, Niterói / Brazil

Key words: Colored spanning trees, acyclic graph, c-edge-colored graph, nonapproximability bounds, polynomial algorithms, NP-completeness.

1 Introduction

The study of problems modeled by edge-colored graphs has given rise to important developments during the last few decades. For instance, the investigation of spanning trees for graphs provide important and interesting results both from a mathematical and an algorithmic point of view (see for instance [1]). From the point of view of applicability, problems arising in molecular biology are often modeled using colored graphs, i.e., graphs with colored edges and/or vertices [6]. Given such an edge-colored graph, original problems translate to extracting subgraphs colored in a specified pattern. The most natural pattern in such a context is that of a proper coloring, i.e., adjacent edges having different colors. Refer to [2,3,5] for a survey of related results and practical applications. Here we deal with some colored versions of spanning trees in edge-colored graphs. In particular, given an edge-colored graph G^c, we address the question of deciding whether or not it contains properly edge colored spanning trees or rooted edge-colored trees with a given pattern.

Formally, let $I_c = \{1, 2, \ldots, c\}$ be a given set of colors, $c \geq 2$. Throughout, G^c denotes an edge-colored simple graph, where each edge is assigned some color $i \in I_c$. The vertex and edge-sets of G^c are denoted $V(G^c)$ and $E(G^c)$, respectively. The *order* of G^c is the number n of its vertices. A subgraph of G^c is said to be *properly edge-colored* if any two of its adjacent edges differ in color. A *tree* in G^c is a subgraph such that its underlying non-colored graph is connected and acyclic. A *spanning tree* is one covering all vertices of G^c. From the earlier definitions, a *properly edge-colored tree* is one such that no two adjacent edges are on a same color. A tree T in G^c with fixed root r is said to be *weakly properly edge-colored* if any path in T, from the root r to any leaf is a properly edge-colored one. To facilitate discussions, in the sequel a properly edge-colored (weakly properly edge-colored) tree will be called a *strong* (*weak*) *tree*. Notice that in the case of weak trees, adjacent edges may have the same color, while this may not happen for strong trees. When these trees span the vertex set of G, they are called *strong spanning tree* (SST) and *weak spanning tree* (WST).

Here we prove that the problems of finding SST and WST in colored graphs are both NP-complete. The problem of SST remains NP-complete even when restricted to the class of edge-colored complete graphs. We present nonapproximability results by considering the optimization versions of these problems. We provide polynomial time algorithms for these problems on the important class of colored acyclic graphs, i.e. graphs without properly edge-colored cycles. We also present an interesting graph theoretic characterization of colored complete graphs which have SSTs.

2 NP-completeness and nonapproximability

The SST problem is NP-complete for G^c if c is a constant, because it generalizes the degree-constrained spanning tree problem, which extends the Hamiltonian path problem. Here, the degree constraint of a node v is the number of different colors used on its incident edges. The next result is a stronger one, and is proved using a kind of self-reduction from the SST problem on a constant number of colors.

Theorem 2.1 *The* SST *is NP-complete even for* $c = \Omega(n^2)$.

The hardness result for WST stated below is obtained by a reduction from the 3-SAT problem.

Theorem 2.2 *Given a 2-edge colored graph* $G^c = (V, E^c)$ *and a specified vertex* r *of* V, *it is NP-complete to determine if* G^c *has a* WST *rooted at* r.

We view the optimization versions of these problems as finding the corresponding trees covering as many vertices as possible. The following results on nonapproximability bounds are obtained by the gap-reduction technique using the MAX-3-SAT problem.

Theorem 2.3 *The maximum weighted tree* (MWT) *problem is nonapproximable within* $63/64 + \epsilon$ *for* $\epsilon > 0$ *unless* $P = NP$.

Theorem 2.4 *The maximum strong tree* (MST) *problem is nonapproximable within* $55/56 + \epsilon$ *for* $\epsilon > 0$ *unless* $P = NP$.

3 Colored trees in acyclic edge-colored graphs

In this section, we present results demonstrating that the SST and WST problems can be solved efficiently when restricted to the class of edge colored-acyclic graphs. We present a proof sketch and an algorithm for the SST problem on colored acyclic complete graphs. The case of SST on general colored acyclic graphs is similar, but more involved and appears in a longer version of the paper. We do not provide the details of the WST problem either, due to space constraints.

An important tool we use is a theorem due to Yeo ([7],[4]), which states that every colored acyclic graph has a vertex v, such that the edges between any component $\mathcal{C}_i$ of $G \setminus v$ and v are monochromatic. We call such a vertex a *yeo-vertex*. If in addition, the colors of the edges between v and the various components obtained by deleting it are all distinct, we call it a *rainbow yeo-vertex*. It is easy to see that a colored acyclic graph with a nonrainbow yeo-vertex has no SST.

For the rest of this section, we assume we are dealing with colored acyclic complete graphs (K_n^c-acyclic). We compute a partial ordering of the vertices and construct the SST by incorporating the vertices in the reverse of this order. The first *block* consists of *all* the yeo-vertices of the graph. They induce a monochromatic clique and the edges between this group of vertices and the rest of the graph are also monochromatic with the same color. We repeat this procedure iteratively, by considering the residual (also) acyclic complete graph obtained by deleting these vertices from the original graph. The second block is also monochromatic but with a *different* color.

We use k to denote the number of blocks in the above partial order and the blocks themselves are denoted $\mathcal{B}_1, \ldots, \mathcal{B}_k$. We use c_i to denote the associated color of block $\mathcal{B}_i$. Recall that the color associated with successive blocks differ. We denote the total number of vertices in the blocks $\mathcal{B}_i, \ldots, \mathcal{B}_k$ by t_i and the number of such vertices whose associated color is l by t_i^l. We now state a lemma, which given an acyclic edge colored complete graph determines whether or not it contains an SST.

Lemma 3.1 (SST-Complete Acyclic) *An acyclic edge colored complete graph has an SST iff*

(1) Last block $\mathcal{B}_k$ has two vertices, and
(2) for each $i < k$,
- *IF block $\mathcal{B}_i$ has the same color as the last block $\mathcal{B}_k$, THEN* $t_i^{c_i} - 2 \leq \frac{t_i}{2}$.
- *ELSE* $t_i^{c_i} \leq \frac{t_i}{2}$.

We now describe our algorithm to construct the SST. It is based on the previous lemma. Its running time is $O(n^2)$, as it can be implemented by modifying the basic Breadth-First-Search (BFS) procedure.

Algorithm 1. SST for K_n^c-acyclic

```
compute the order described above
if last block B_k has more than two vertices then return "No SST"
if last block B_k has two vertices then connect the two vertices of B_k to
   get an initial Strong Tree
for i = k − 1 to 1 do
   if condition 2 of Lemma 3.1 is true then
      join the vertices of B_i as leaves, to distinct vertices already incorpo-
      rated in the tree which have not used an edge of color c_i in the partial
      strong tree obtained in the previous iteration.
   else
      return "NO SST"
   end if
end for
return the SST
```

To conclude this section, we now state our more general result.

Theorem 3.2 *The* SST *and* WST *problems can be solved efficiently for acyclic colored graphs.*

4 Properly edge-colored spanning trees in edge-colored complete graphs

The SST problem remains hard even when stringently restricted, as the following result states. The hardness is proved by a reduction from the SST problem in general graphs.

Theorem 4.1 *The* SST *is NP-complete for complete graphs K_n^c, colored with $|c| \geq 3$ colors.*

Observe, that for the case $c = 2$, the SST problem reduces to the Hamiltonian Path problem, which is known to be polynomial [3]. Notice also that the WST problem is trivial in K_n^c as any spanning star is a WST. Concerning SST, we provide below a graph-theoretic characterization for edge-colored complete graphs K_n^c which have SSTs. This characterization is interesting from a mathematical point of view, but the implied conditions cannot be computed in polynomial time, in view of the hardness result above.

Theorem 4.2 *Assume that the vertices of K_n^c are covered by a strong tree T and a set of properly edge-colored cycles, say $C_1, C_2 \cdots, C_k$ all these components being pairwise vertex-disjoint in K_n^c. Then K_n^c has a strong spanning tree.*

References

[1] R. K. Ahuja, T. L. Magnanti, J. B. Orlin, Network Flows: Theory, Algorithms and Applications, *Prentice Hall*, 1993.

[2] J. Bang-Jensen, G. Gutin, Digraphs: Theory, Algorithms and Applications, *Springer*, 2002.

[3] A. Benkouar, Y. Manoussakis, V. T. Paschos, R. Saad, On the Complexity of Some Hamiltonian and Eurelian Problems in Edge-colored Complete Graphs *RAIRO - Operations Research*, **30**, 417-438, 1996.

[4] J. Grossman and R. Häggkvist, properly edge-coloredCycles in Edge-Partioned Graphs, *Journal of Combinatorial Theory, Series B*, **34** (1983) 77-81.

[5] Y. Manoussakis, properly edge-coloredPaths in Edge-Colored Complete Graphs, *Discrete Applied Mathematics*, **56** (1995) 297-309..

[6] P. A Pevzner, DNA Physical Mapping and Properly Edge-colored Eurelian Cycles in Colored Graphs, *Algorithmica*, **13** (1995) 77-105 .

[7] A. Yeo, A Note on Alternating Cycles in Edge-colored Graphs, *Journal of Combinatorial Theory, Series B* **69** (1997) 222-225 .